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**Testing Equilibrium Behaviour  
At First-Price, Sealed-Bid  
Auctions With Discrete Bid  
Increments**

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# Testing Equilibrium Behaviour At First-Price, Sealed-Bid Auctions With Discrete Bid Increments<sup>\*</sup>

Harry J. Paarsch<sup>†</sup>, Jacques Robert<sup>‡</sup>

## Résumé / Abstract

Dans le cadre du modèle d'enchères à valeurs privées et indépendantes, nous dérivons les implications empiriques du comportement d'équilibre dans les enchères à enveloppe scellée au premier prix, lorsque les incréments sont discrets. Bien que les comportements d'équilibre diffèrent lorsque les mises prennent des valeurs discrètes plutôt que continues, ces différences disparaissent typiquement comme les incréments tendent vers zéro. En pratique, les mises dans les enchères prennent des valeurs discrètes. De plus, ceci peut simplifier les calculs nécessaires pour estimer et tester les modèles d'enchères. Notre approche s'insère dans le cadre d'un modèle d'inférence incomplet et nous permet de tester si les données sont cohérentes avec un comportement d'équilibre dans le cadre d'un modèle symétrique aussi bien qu'asymétrique. Nous démontrons l'utilité de l'approche en l'appliquant à des données générées à partir d'expériences en laboratoire.

**Mots clés** : inférence dans les enchères, mises à valeur discrètes, enchères asymétriques.

*Within the independent private-values paradigm, we derive the equilibrium implications of purposeful bidding behaviour at single-unit, first-price, sealed-bid auctions when discrete increments are imposed on bidding. While equilibrium purposeful behaviour with discrete bid increments is different from that which would obtain were continuous variation in bids permitted, these differences typically disappear as the bid increments go to zero. But discrete bid increments are a common feature at many real-world auctions. Moreover, their presence can simplify computation when estimating and testing auction models. Our approach fits within a model of incomplete inference and allows us to test sequentially for symmetric, equilibrium purposeful behaviour as well as asymmetric, equilibrium purposeful behaviour. We demonstrate the utility of the approach by applying it to data from laboratory experiments.*

**Keywords:** *inference in auctions, discrete bidding, asymmetric auctions.*

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## 1. Introduction

During the last four decades, economists have made considerable progress in understanding the theoretical structure of strategic behaviour under market mechanisms, such as auctions, when a small number of potential participants exists; see Krishna (2002) for a comprehensive book-length survey of progress. During the last fifteen years, two distinct empirical strategies designed to take these theoretical models to either experimental or field data have been proposed; see Hendricks and Paarsch (1995) for a review.

One analytic device, commonly used to describe bidder motivation at auctions, is a continuous random variable which represents individual-specific heterogeneity in valuations. The conceptual experiment involves each potential bidder's receiving an independent draw from a distribution of valuations. Conditional on this random variable, the economic actors are assumed to act purposefully, maximizing either expected profit or the expected utility of profit from winning the auction. An assumption then made for computational parsimony is that agents choose their bids continuously when maximizing their expected objective function. Of course, this seemingly innocuous and certainly natural assumption is made so that the calculus can be used to characterize optimal behaviour. Another frequently-made assumption is that agents are *ex ante* symmetric, their independent draws coming from the same distribution of valuations, an assumption that then allows one to aggregate across all potential economic actors and to focus on one "representative" first-order condition when describing optimal behaviour. Finally, researchers typically impose a notion of equilibrium, such as Bayes-Nash, to close the model.

At many real-world auctions, economic agents are required to submit bids in discrete increments. In such cases, equilibrium purposeful behaviour is different from that which would obtain in a continuous-variation model. Also, in many economic environments, asymmetries in valuations across bidders of different types are often important, making the assumption of symmetry untenable. Finally, a notion of equilibrium is often used to identify the data-generating process (DGP) of the model (in the Hood-Koopmans econometric sense), so it is then impossible to test such a notion without recourse to additional assumptions; *e.g.*, a functional-form assumption concerning the distribution from which the valuations are drawn.

While each of the three maintained assumptions described above is made for computational parsimony and tractability, it is of considerable interest to have a strategy to decide whether individually and together they are, in fact, an accurate characterization of the phenomena they are used to model. This turns out to be extremely difficult to carry out computationally in the continuous-variation model, except for very simple examples. In this paper, we hope to make some progress on this problem by relaxing the assumption of continuous choice, assuming instead that potential participants at auctions are constrained to bid in fixed and known discrete increments.

By focussing on observed market behaviour, specifically non-participation as well as the bids tendered at single-unit, first-price, sealed-bid auction, we then characterize the conditions under which the observed actions of potential bidders are consistent with the notion of symmetric, Bayes-Nash equilibrium behaviour. We then go on to specify conditions sufficient to test for just Bayes-Nash equilibrium behaviour, relaxing the symmetry assumption. Finally, we admit the presence of differential risk aversion among potential bidders. We demonstrate the utility of our proposed approach by applying it to data from laboratory experiments.

Ours is a model of incomplete inference studied first in the econometrics literature by Horowitz and Manski (1995) and then used to interpret field data concerning oral, ascending-price (English) auctions by Haile and Tamer (2003). Haile and Tamer (2003) concerned themselves with the “continuous button” model of an English auction, noting that observed data from real-world oral, ascending-price auctions often occur in jumps, do not vary continuously. Our concept of equilibrium, Bayes-Nash, is different from that of Haile and Tamer (2003) who used the concept of dominance. Moreover, whereas Haile and Tamer (2003) provide no test of equilibrium purposeful behaviour, but rather use the theory as an identifying assumption, we provide explicit tests against different alternatives which may be non-nested. Like Haile and Tamer (2003), we note that the discrete nature of bidding prevents the econometrician from recovering an estimate of the “exact” distribution of latent heterogeneity. And, as in Haile and Tamer (2003), we note that this lack of precision changes the nature of the optimal mechanism-design problem.

Our research is distinct from that of Athey and Haile (2002) who considered the identification of standard (continuous-variation) models of auctions under alternative data sampling schemes, but the work is complementary to that of Guerre, Perrigne, and Vuong (2000) as well as Campo, Guerre, Perrigne, and Vuong (2000) who investigated non-parametric identification, estimation, and testing in continuous-variation, symmetric, first-price, sealed-bid auction models within the independent private-values paradigm (IPVP), without and with risk aversion.

## 2. Motivating Empirical Example

Consider an empirical worker who has collected a sample of data concerning a sequence of  $T$  first-price, sealed-bid auctions at which single, identical objects are sold independently to  $n$  potential buyers. The rules of the auction require that bidders submit sealed tenders from a countable, finite set of discrete bids. The empirical worker summarizes these bidding data into counts where  $m_i$  denotes the number of times a bid  $k_i$  was observed where  $i = 0, 1, \dots, K$ ;  $m_0$  denotes the number of times that a bidder declined to participate, did not bid. Note that  $m_0$  equals  $(nT - \sum_{i=1}^K m_i)$ . The vector  $\mathbf{m}$ , which equals  $(m_0, m_1, \dots, m_K)$ , collects these.

Now, under the assumption of independence and for some true model, the

probability of having observed  $\mathbf{m}$  is given by the multinomial distribution

$$g(\mathbf{m}|\boldsymbol{\pi}^0) = \frac{(nT)!}{\prod_{i=0}^K m_k!} \prod_{i=0}^K (\pi_i^0)^{m_i} \quad (2.1)$$

where the probability of bid  $k_i$  under the true model is denoted  $\pi_i^0$  and where the vector  $\boldsymbol{\pi}^0$  collects the  $\pi_i^0$ s, equalling  $(\pi_0^0, \pi_1^0, \dots, \pi_K^0)$ . Given  $\boldsymbol{\pi}^0$ , calculating the probability that the observed data  $\mathbf{m}$  came from (2.1) is straightforward. We are interested in deciding whether the true model is “Bayes-Nash consistent” (BNC). Our use of BNC is closest to the use of “rationalizable” by Campo, Guerre, Perrigne, and Vuong (2000). The term BNC is different from other ways in which the term “rationalizable” has been used in economics, particularly by theorists; see, for example, Dekel and Wolinsky (2001), Battigalli and Siniscalchi (2002), and Cho (2002). To appreciate our usage of BNC, we turn next to the specification of a theoretical model in which the term has content.

### 3. Theoretical Model

We consider a theoretical model in which  $n$  potential bidders vie for the right to buy a single object at a first-price, sealed-bid auction. At this auction, agents must submit bids in discrete increments. Let  $\mathcal{K}$ , equal  $\{k_0, k_1, k_2, \dots, k_K, k_{K+1}\}$ , denote the set of possible bids submitted. Here  $k_{i+1}$  equals  $(k_i + \Delta_i)$  for positive increments  $\Delta_i$  and all  $i = 1, \dots, K$ . We denote the lowest admissible bid, the reserve price, by  $k_1$  and the highest observed bid, but not necessarily the highest admissible one, by  $k_K$ . In principle, a bidder could submit  $k_{K+1}$  which equals  $(k_K + \Delta_K)$ . For non-participants, we denote the null bid by  $k_0$  which, without loss of generality, we normalize to be zero. When the increments are the same, we suppress the subscript and just refer to the increment as  $\Delta$ . In the event of ties, the object is allocated at random to one of the tied bidders. Specifically, if  $\ell$  bidders each submit the same highest bid  $k_j$ , then one is chosen at random to be the winner with probability  $(1/\ell)$ .

Within our model, potential bidders have valuations that are assumed initially to be independently and identically distributed draws of a continuous random variable  $V$  having cumulative distribution function  $F(v)$ , probability density function  $f(v)$ , and support on the interval  $[v, \bar{v}]$ ; *i.e.*, we work within the IPVP.

Our theoretical model is developed in terms of observables; *viz.*, the number of potential bidders  $n$  as well as the distributions of non-participation and observed bids. To this end, we denote the probability of observing bid  $k_i$  by  $\pi_i$ ;  $\pi_0$  denotes the probability of not participating at the auction. We collect the  $\pi_i$ s in the vector  $\boldsymbol{\pi}$  which equals  $(\pi_0, \pi_1, \dots, \pi_K)$ ;  $\boldsymbol{\pi}$  denotes the probability distribution over  $\mathcal{K}$  where  $\sum_{i=0}^K \pi_i$  equals one. We assume that, like  $n$  and  $F(v)$ ,  $\boldsymbol{\pi}$  is common knowledge to the potential bidders. We introduce  $\Pi_i$ , the cumulative distribution function, which

equals  $\sum_{j=0}^i \pi_j$ , the probability that a bidder bids  $k_i$  or less, and collect all of these in the vector  $\mathbf{\Pi}$  which equals  $(\pi_0, \Pi_1, \dots, \Pi_{K-1}, 1)$ .

We assume that a representative potential bidder of type  $v$  seeks to maximize the expected profit from winning the auction; *i.e.*,

$$\max_{\langle k_t \in \mathcal{K} \rangle} (v - k_t) \Pr(\text{winning} | k_t).$$

For a representative bidder, we denote by  $\Gamma_i$  the equilibrium probability of winning if  $k_i$  is submitted; these probabilities are collected in  $\mathbf{\Gamma}$  equal  $(\Gamma_0, \Gamma_1, \dots, \Gamma_K, \Gamma_{K+1})$ . Using  $\mathbf{\pi}$ , we seek to calculate the elements of the vector  $\mathbf{\Gamma}$ . We know some of the  $\Gamma_i$ s already. For example, if some one does not participate at the auction, then the probability of his winning is zero, so  $\Gamma_0$  equals zero. Also, if some one bids more than the observed maximum, then he is sure to win, so  $\Gamma_{K+1}$  equals one. For the remaining  $\Gamma_i$ s, one can verify directly that the following constitutes a symmetric, Bayes-Nash equilibrium of the auction game:

$$\Gamma_i = \frac{(\Pi_i)^n - (\Pi_{i-1})^n}{n(\Pi_i - \Pi_{i-1})} \quad \forall i = 1, 2, \dots, K. \quad (3.1)$$

The numerator of (3.1) is the probability that the highest bid is exactly equal to  $k_i$ , while the denominator is the expected number of potential bidders submitting bid  $k_i$ .

We now consider a representative bidder's best response when confronted by  $\mathbf{\Gamma}$ . For valuation  $v$ , it is optimal to bid  $k_i$  when the following inequalities hold:

$$(v - k_i)\Gamma_i \geq (v - k_j)\Gamma_j \quad \forall j \neq i. \quad (3.2)$$

In words, the expected profit from bid  $k_i$  weakly exceeds that for any alternative bid  $k_j$ . Of course, this is just the definition of  $k_i$ 's being optimal. Obviously, this set of inequalities is the discrete analogue to the equilibrium first-order condition for expected-profit maximization in the continuous-variation model which takes the form of the following ordinary differential equation in the strategy function  $\sigma(v)$ :

$$\sigma'(v) + \sigma(v) \frac{(n-1)f(v)}{F(v)} = v \frac{(n-1)f(v)}{F(v)}. \quad (3.3)$$

When the reserve price is  $r$ , (3.3) has the following convenient solution:

$$\sigma(v) = v - \frac{\int_r^v F(u)^{n-1} du}{F(v)^{n-1}} \quad (3.4)$$

which, except perhaps for the integral in the numerator of the right-most term on the right-hand side of (3.4), is much more compact to deal with computationally than the set of inequalities presented by (3.2).

#### 4. Bayes-Nash Consistent Behaviour

Given the expected-profit maximizing, Bayes-Nash equilibrium behaviour characterized in the previous section, what restrictions does this behaviour impose on the DGP? We introduce the notion of BNC behaviour, a term whose usage is similar to the way in which Campo *et al.* (2000) use the term “rationalizable” within the continuous-variation model. Since our theoretical framework is slightly different from Campo *et al.*, we state separately our definition below.

**Definition 1:** A vector  $\boldsymbol{\pi}$  over  $\mathcal{K}$  is Bayes-Nash consistent if there exists a (Bayes-Nash) equilibrium that generates  $\boldsymbol{\pi}$  as an equilibrium distribution of bids.

**Theorem 1:** A vector  $\boldsymbol{\pi}$  over  $\mathcal{K}$  is Bayes-Nash consistent, if and only if, for all  $i$  such that  $\pi_i > 0$ ,

$$\frac{k_i\Gamma_i - k_t\Gamma_t}{\Gamma_i - \Gamma_t} \leq \frac{k_i\Gamma_i - k_s\Gamma_s}{\Gamma_i - \Gamma_s} \quad \forall 1 \leq t < i < s \leq K + 1 \quad (4.1)$$

where  $\Gamma_i$  is defined in terms of  $\boldsymbol{\pi}$  by (3.1).

**Proof:** The expression

$$v_{t,i} \equiv \frac{k_i\Gamma_i - k_t\Gamma_t}{\Gamma_i - \Gamma_t}$$

corresponds to the type which is indifferent between bidding  $k_i$  and  $k_t$ . Denote

$$\underline{v}_i \equiv \max_{t < i} \left[ \frac{k_i\Gamma_i - k_t\Gamma_t}{\Gamma_i - \Gamma_t} \right]$$

$$\bar{v}_i \equiv \min_{s > i} \left[ \frac{k_i\Gamma_i - k_s\Gamma_s}{\Gamma_i - \Gamma_s} \right]$$

When the conditions of Theorem 4.1 hold,  $\underline{v}_i$  is less than or equal to  $\bar{v}_i$  for all  $i$  such that  $\pi_i$  is strictly positive. All types  $v$  that are greater than or equal to  $\underline{v}_i$  will prefer to bid  $k_i$  rather than any lower price; all types  $v$  that are less than or equal to  $\bar{v}_i$  will prefer to bid  $k_i$  rather than any higher price. It follows that, for all  $v \in [\underline{v}_i, \bar{v}_i]$ , it is optimal to bid  $k_i$ . We need to assign a probability that  $v$  lies in  $[\underline{v}_i, \bar{v}_i]$ , equal  $\pi_i$ . If  $\underline{v}_i$  equals  $\bar{v}_i$ , then we assume a mass point at  $\bar{v}_i$  in the distribution of valuations. The corresponding  $F(\cdot)$  is constructed so that the observed distribution of bids is consistent with the strategy of the bidder and the distribution of types. Each participant bids optimally, given the distribution of others' bids. Now, when the conditions of Theorem 4.1 fail to hold and  $\underline{v}_i$



exceeds  $\bar{v}_i$ , then no  $v$  exists for which it is optimal to bid  $k_i$ . Hence, we must have  $\pi_i$  equal to zero.

Verifying that the vector of sample bid proportions  $\hat{\mathbf{p}}$ , which equals  $(\mathbf{m}/T/n)$ , is BNC simply involves verifying that the  $\hat{\Gamma}_i$ s, the estimated  $\Gamma_i$ s which are calculated using  $\hat{\mathbf{p}}$  in place of  $\boldsymbol{\pi}$  according to equation (3.1), satisfy the inequalities in (4.1) of Theorem 1. This step is equivalent to verifying that the estimated  $\xi(v)$  function, in the notation of Guerre *et al.* (2000), is a monotonically increasing function of the valuation  $v$ . In our model, this step requires less computational effort because the bids come in discrete increments, which happen to be a reality in many problems.<sup>1</sup> Note that when  $\hat{\mathbf{p}}$  is BNC, one strategy for estimating the p-value, at least asymptotically under the null, simply involves simulating draws  $\mathbf{S}_{-0}$  from the multivariate normal distribution; *i.e.*,

$$\mathbf{S}_{-0} \sim \mathcal{N}[\hat{\mathbf{p}}_{-0}, \boldsymbol{\Sigma}(\hat{\mathbf{p}}_{-0})]$$

where  $\hat{\mathbf{p}}_{-0}$  is  $\hat{\mathbf{p}}$  without  $\hat{p}_0$  and

$$\boldsymbol{\Sigma}(\hat{\mathbf{p}}_{-0}) = \begin{pmatrix} \hat{p}_1(1 - \hat{p}_1) & -\hat{p}_1\hat{p}_2 & \dots & -\hat{p}_1\hat{p}_K \\ -\hat{p}_1\hat{p}_2 & \hat{p}_2(1 - \hat{p}_2) & \dots & -\hat{p}_2\hat{p}_K \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{p}_1\hat{p}_K & -\hat{p}_2\hat{p}_K & \dots & \hat{p}_K(1 - \hat{p}_K) \end{pmatrix} \quad (4.2)$$

and then keeping track of the proportion of rejections according to (4.1) when  $\mathbf{s}$ , which is  $\mathbf{s}_{-0}$  augmented by  $s_0$  which equals  $(1 - \sum_{i=1}^K s_i)$  is used instead of  $\boldsymbol{\pi}$ .

Note that  $\boldsymbol{\pi}$  lives in the simplex  $\mathcal{S}^K$ . In that simplex, we denote the set of all points that are BNC by  $\mathcal{C}$  (for ‘‘consistent’’) which is a subset of  $\mathcal{S}^K$ . Of course, the reader might ask: Can  $\mathcal{C}$  be empty? No. A uniform distribution on  $\boldsymbol{\pi}$ , the point in the middle of  $\mathcal{S}^K$  which equals  $([1/(K+1)], [1/(K+1)], \dots, [1/(K+1)])$ , will always be BNC. For a given distribution  $F(v)$  and under the sort of equilibrium purposeful behaviour assumed in section 3, the ‘‘true’’ model  $\boldsymbol{\pi}^0$  will live in  $\mathcal{C}$ . Note too that other candidate  $\boldsymbol{\pi}$ s can also live in  $\mathcal{C}$ .

When  $\hat{\mathbf{p}}$  strays out of  $\mathcal{C}$ , two reasons can exist: first, sampling error or, second, agents are not behaving according to theory, for whatever cause. We need to assess the relative importance of these two explanations and then provide a strategy to disentangle the potential reasons why agents are not behaving according to the theory. Obviously, without additional information, we cannot ascertain what the unknown  $\boldsymbol{\pi}^0$  is; we can only try to estimate it.

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<sup>1</sup> Using the Guerre *et al.* (2000) approach, one would first find the optimal bandwidth, next kernel-smooth the bid distribution, then construct an estimate of  $\xi(v)$ , and finally check to see that  $\hat{\xi}(v)$ , the estimate of  $\xi(v)$ , is monotonically increasing in  $v$ .

Now, the statistical sampling theory for the multinomial distribution is extremely well developed, so we can deal with sampling variation in a straightforward manner, provided we can calculate the probability of observing a  $\hat{\mathbf{p}}$  not contained in  $\mathcal{C}$ , given a  $\boldsymbol{\pi}$  contained in  $\mathcal{C}$ .

To this end, we seek some of  $\mathcal{C}$ 's properties. First, the set  $\mathcal{C}$  is quite large, relatively speaking. For example, when  $K$  is small, one can derive, by brute-force calculation, a set of inequalities that characterizes  $\mathcal{C}$ . If  $n$  is two and  $K$  is three, then the following defines  $\mathcal{C}$ :

$$\frac{(2 - \pi_1 - 2\pi_2 - 2\pi_3)(\pi_2 + \pi_3)}{(\pi_1 + \pi_2)} \leq 1 \quad (4.3)$$

in the simplex  $\mathcal{S}^3$ . (The reader should note that when  $n$  is two and  $K$  is two, the set  $\mathcal{C}$  equals the simplex  $\mathcal{S}^2$ . This obtains because any behaviour can be rationalized at either of the endpoints and, with a  $K$  of two, basically, only endpoints exist so no other constraints on the process exist.) In dimensions for  $K$  that are higher than three, characterizing  $\mathcal{C}$  is computationally arduous. Also, trying to bound  $\mathcal{C}$  is difficult because the set can be non-convex.<sup>2</sup> Thus, the approximation error associated with any bounding strategy is potentially quite large. What to do?

## 5. Testing BNC Behaviour

From a computational perspective, characterizing  $\mathcal{C}$  is extremely difficult. But we need  $\mathcal{C}$  because, when  $\hat{\mathbf{p}}$  is not BNC, we want to estimate the probability of getting  $\hat{\mathbf{p}}$ , given the “nearest” BNC point. This forms the basis of our testing procedure. Now, the problems of characterizing  $\mathcal{C}$  and finding a point in  $\mathcal{C}$  closest to  $\hat{\mathbf{p}}$  are obviously related. However, because these are conceptually distinct problems, we break up discussing them, examining each in series.

### 5.1. Distance from $\hat{\mathbf{p}}$ to $\mathcal{C}$

How does one calculate distance in  $\mathcal{S}^K$  to define the nearest point? One possibility is just the Euclidean distance  $\|\hat{\mathbf{p}} - \boldsymbol{\pi}\|_2$ . Estimating this distance involves solving the following constrained optimization problem:

$$\min_{\langle \boldsymbol{\pi} \rangle} \|\hat{\mathbf{p}} - \boldsymbol{\pi}\|_2 \quad \text{subject to} \quad \boldsymbol{\pi} \in \mathcal{C} \subset \mathcal{S}^K.$$

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<sup>2</sup> To see this, impose equality in (4.3) and write  $\pi_1$  as a function of  $\pi_2$  and  $\pi_3$ . Straightforward differentiation reveals that the derivatives can change sign, when either  $\pi_2$  (or  $\pi_3$ ) are held constant while  $\pi_3$  (or  $\pi_2$ ) are varied, implying that the set is non-convex.

Now, the variance of  $\hat{p}_i$  depends on the relative value of  $\pi_i$ . To adjust for this, one might use a second possibility; *viz.*, minimizing the Mahalanobis norm (see Rao 1973, p. 542), which involves solving the following constrained optimization problem:

$$\min_{\langle \boldsymbol{\pi}_{-0} \rangle} (\hat{\mathbf{p}}_{-0} - \boldsymbol{\pi}_{-0})^\top [\boldsymbol{\Sigma}(\boldsymbol{\pi}_{-0})]^{-1} (\hat{\mathbf{p}}_{-0} - \boldsymbol{\pi}_{-0}) \quad \text{subject to} \quad \boldsymbol{\pi} \in \mathcal{C} \subset \mathcal{S}^K$$

where  $\boldsymbol{\Sigma}(\boldsymbol{\pi}_{-0})$  is defined in (4.2). We propose a third alternative, seeking instead to maximize the empirical likelihood of having observed  $\mathbf{m}$  subject to  $\boldsymbol{\pi}$ 's being in  $\mathcal{C}$ :

$$\max_{\langle \boldsymbol{\pi} \rangle} g(\boldsymbol{\pi} | \mathbf{m}) \quad \text{subject to} \quad \boldsymbol{\pi} \in \mathcal{C} \subset \mathcal{S}^K$$

or, alternatively, under a monotonic (logarithmic) transformation, ignoring the constant term,

$$\max_{\langle \boldsymbol{\pi} \rangle} \sum_{i=0}^K m_i \log(\pi_i) \quad \text{subject to} \quad \boldsymbol{\pi} \in \mathcal{C} \subset \mathcal{S}^K. \quad (5.1)$$

We chose this objective function because we want to use the likelihood-ratio test to decide whether  $\hat{\mathbf{p}}$  is “significantly” different from a BNC outcome and because (5.1) is related to the Kullback-Leibler distance. We hope to exploit this relationship and to integrate our model-testing work with that of Vuong (1989).

To see that (5.1) is related to the Kullback-Leibler distance, transform

$$\sum_{i=0}^K m_i \log(\pi_i), \quad (5.2)$$

linearly, by multiplying it by  $(1/nT)$  and then subtracting  $\sum_{i=0}^K \hat{p}_i \log \hat{p}_i$  to get

$$\sum_{i=0}^K \frac{m_i}{nT} \log(\pi_i) - \sum_{i=0}^K \hat{p}_i \log \hat{p}_i = \sum_{i=0}^K \hat{p}_i \log \left( \frac{\pi_i}{\hat{p}_i} \right).$$

Now,

$$\sum_{i=0}^K \hat{p}_i \log \left( \frac{\hat{p}_i}{\pi_i} \right) = - \sum_{i=0}^K \hat{p}_i \log \left( \frac{\pi_i}{\hat{p}_i} \right) \quad (5.3)$$

is the Kullback-Leibler distance, so maximizing (5.2) is equivalent to minimizing (5.3).<sup>3</sup>

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<sup>3</sup> Yet a fourth alternative would be the following:

$$\min_{\langle \boldsymbol{\pi} \rangle} - \sum_{i=0}^K \pi_i \log \left( \frac{\hat{p}_i}{\pi_i} \right) \quad \text{subject to} \quad \boldsymbol{\pi} \in \mathcal{C} \subset \mathcal{S}^K.$$

## 5.2. An Alternative Representation of $\mathcal{C}$

Under any of the above distance alternatives, even if characterizing  $\mathcal{C}$  could be done quickly, the fact that  $\mathcal{C}$  is typically non-convex makes the above constrained optimization problem difficult to solve accurately. In an effort to reduce computational problems, we adopt an heuristic made popular by Murtagh and Saunders (1983). In particular, we seek a transformation of the above problem in which the difficulties in characterizing  $\mathcal{C}$  are embedded in the objective function, leaving the newly-transformed, constrained-optimization problem with a choice set that is “easy” to deal with computationally, in our case, convex. Of course, such a transformation can often take a well-behaved objective function (as the logarithm of the likelihood function is in the multinomial case, being a strictly concave function of the relevant  $\pi$ s) and introduce local optima. This complication can potentially affect the size of our testing procedure, but we shall discuss this later. Below, we describe the reasoning behind the particular transformation we use as well as the specifics of implementing it.

Since  $V$  is a continuous random variable, we know that an interval will exist in which it will be optimal to bid  $k_i$ . Because bids are monotonic, the interval, which can be empty, is connected.<sup>4</sup> When  $\pi_i$  is strictly positive, we know that the interval of “type”s for which the best response is to bid  $k_i$  must be non-empty, otherwise one could not justify the bid  $k_i$  having obtained with positive probability in the first place.

For a representative bidder, let  $v_i$  denote the type which is indifferent between bidding  $k_i$  and  $k_{i+1}$ . We define  $v_0$  to be that bidder type who is indifferent between not participating and bidding the reserve price at the auction. Hence  $v_0$  equals  $k_1$ , the reserve price. We introduce  $\mathcal{V}(\mathcal{K})$  as the set of all vectors  $\mathbf{v}$  equal  $(v_0, v_1, \dots, v_K)$  which satisfy the following constraints:

$$\begin{aligned} k_1 &= v_0 \\ k_{i+1} &\leq v_i \quad i = 1, \dots, K \\ v_0 &\leq v_1 \leq v_2 \leq \dots \leq v_K \\ v_K &\leq \bar{v}. \end{aligned}$$

For a given  $\mathcal{K}$  one can see, by direct inspection, that the set  $\mathcal{V}(\mathcal{K})$  is compact and convex. Moreover, any  $\mathbf{v} \in \mathcal{V}(\mathcal{K})$  gives rise to a unique  $\boldsymbol{\pi} \in \mathcal{C}$ .

## 5.3. Mechanics of Computation

We start by assuming no complications, which are typically introduced by the presence of  $\pi_i$ s that equal zero, and then introduce additional elements of numerical subtlety.

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<sup>4</sup> We assert this without proof; a proof will be supplied later.

For a candidate  $\tilde{\mathbf{v}} \in \mathcal{V}(\mathcal{K})$ , one can calculate the  $\tilde{\Gamma}$  consistent with  $\tilde{\mathbf{v}}$  as follows: first, consider the equal-expected-profit condition between adjacent pairs of  $k_i$  and  $k_{i+1}$  which must hold if  $\tilde{v}_i$  is to be an indifference point. Thus,

$$(\tilde{v}_i - k_i)\tilde{\Gamma}_i = (\tilde{v}_i - k_{i+1})\tilde{\Gamma}_{i+1} \quad i = K, K-1, \dots, 1,$$

so

$$\tilde{\Gamma}_i = \frac{(\tilde{v}_i - k_{i+1})}{(\tilde{v}_i - k_i)}\tilde{\Gamma}_{i+1} \quad i = K, K-1, \dots, 1$$

where  $\tilde{\Gamma}_{K+1}$  equals one and  $\tilde{\Gamma}_0$  equals zero. In general, however, one must also entertain the possibility of alternative bids above  $k_{i+1}$  being optimal, so

$$\tilde{\Gamma}_i = \max_{j>i} \left[ \frac{(\tilde{v}_i - k_j)}{(\tilde{v}_i - k_i)}\tilde{\Gamma}_j \right].$$

Now, for a symmetric game,  $\tilde{\boldsymbol{\pi}}$  is defined implicitly by

$$\tilde{\Gamma}_i = \frac{(\tilde{\Pi}_i)^n - (\tilde{\Pi}_{i-1})^n}{n(\tilde{\Pi}_i - \tilde{\Pi}_{i-1})} \quad \forall i = K, K-1, \dots, 1.$$

But  $\Pi_K$  equals one while the above expression is strictly monotonic in  $\tilde{\Pi}_{i-1}$  for all  $\tilde{\Pi}_{i-1}$  between zero and  $\tilde{\Pi}_i$  when  $\tilde{\Pi}_i$  is strictly positive. Hence, for all  $\tilde{\Gamma}_i$  such that

$$\frac{(\tilde{\Pi}_i)^n}{n} < \tilde{\Gamma}_i < (\tilde{\Pi}_i)^{n-1},$$

a unique  $\tilde{\Pi}_{i-1}$  exists and is contained in  $[0, \tilde{\Pi}_i]$  which equals  $[0, \tilde{\Pi}_{i-1} + \tilde{\pi}_i]$  where  $\tilde{\pi}_i$  is non-negative. Obviously, once one has  $\tilde{\Pi}$  calculating  $\tilde{\boldsymbol{\pi}}$  is trivial.

The constrained optimization we solve has the following, relatively simple struc-

ture:

$$\min_{\langle \mathbf{v} \rangle} - \sum_{i=0}^K \hat{p}_i \log[\pi_i(\mathbf{v})/\hat{p}_i] \quad \text{subject to}$$

$$\begin{pmatrix} -k_1 \\ k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_{K+1} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\bar{v} \end{pmatrix} \leq \begin{pmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_K \end{pmatrix}$$

where a candidate  $\tilde{\mathbf{v}}$  induces a candidate  $\tilde{\boldsymbol{\pi}}$  which we denote implicitly by

$$\log \tilde{\boldsymbol{\pi}} = \mathbf{h}(\tilde{\mathbf{v}}).$$

Thus, we summarize the mathematical programme in the following notation:

$$\min_{\langle \mathbf{v} \rangle} \mathbf{c}^\top \mathbf{h}(\mathbf{v}) \quad \text{subject to} \quad \mathbf{b} \leq \mathbf{A}\mathbf{v} \leq \mathbf{u} \quad \text{and} \quad \mathbf{v} \in [\underline{v}, \bar{v}]^{K+1}. \quad (5.4)$$

Note that the sparse nature of the constraint matrix  $\mathbf{A}$  makes this problem particularly amenable to solution using the methods of Murtagh and Saunders (1978). In fact, in applications, we use the programme SNOPT to solve this problem numerically; see Gill, Murray and Saunders (1997).<sup>5</sup> SNOPT uses a sequential quadratic programming algorithm that obtains search directions from a sequence of quadratic programming subproblems. Each quadratic-programme subproblem is solved by minimizing a quadratic form of a certain Lagrangian function subject to linear constraints. An augmented Lagrangian merit function is reduced along each search direction to ensure convergence from any starting point. SNOPT is most efficient when only some

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<sup>5</sup> One could also use MINOS as documented by Murtagh and Saunders (1983). We chose SNOPT because it was available while MINOS was not.

of the variables enter nonlinearly, which is not our case as all do, or when the number of active constraints (including simple bounds) is nearly as large as the number of variables, which is our case.

Of course, not all candidate  $\tilde{\mathbf{v}}$ s are consistent with  $\tilde{\boldsymbol{\pi}}$ 's being strictly positive for all elements. This will manifest itself by some

$$\tilde{\Gamma}_i \geq (\tilde{\Pi}_i)^{n-1} \quad (5.5)$$

or

$$\tilde{\Gamma}_i \leq \frac{(\tilde{\Pi}_i)^n}{n}. \quad (5.6)$$

When (5.5) obtains, we know that  $\tilde{\pi}_i$  must be zero, so we set  $\tilde{\Gamma}_i$  to  $(\tilde{\Pi}_i)^{n-1}$  and  $\tilde{\Pi}_{i-1}$  to  $\tilde{\Pi}_i$ . When (5.6) obtains, we set  $\tilde{\Gamma}_i$  to  $[(\tilde{\Pi}_i)^n/n]$  and  $\tilde{\Pi}_{i-1}$  to zero, so  $\tilde{\pi}_i$  equals  $\tilde{\Pi}_i$ . Note that in this case all  $\tilde{\pi}_j$ s are zero for  $j$  less than  $i$ .

One final check must be completed in order to complete our procedure. Let  $\ell$  equal the index  $i$  for which  $\tilde{\pi}_i$  is currently being solved. Denote by  $k_\ell$  the lowest price submitted in the proposed equilibrium. Using only the  $\tilde{\Pi}_i$ s calculated thus far, verify that

$$(\tilde{v}_{\ell+1} - k_{\ell+1}) \frac{(\tilde{\Pi}_{\ell+1})^n - (\tilde{\Pi}_\ell)^n}{n(\tilde{\Pi}_{\ell+1} - \tilde{\Pi}_\ell)} \geq (\tilde{v}_{\ell+1} - k_\ell) \frac{(\tilde{\Pi}_\ell)^{n-1}}{n}.$$

If this is true, then one is done; otherwise one must find the smallest  $j$  which exceeds  $(i + 1)$  such that

$$(\tilde{v}_j - k_j) \frac{(\tilde{\Pi}_j)^n - (\tilde{\Pi}_{j-1})^n}{n(\tilde{\Pi}_j - \tilde{\Pi}_{j-1})} \geq \max \left[ (\tilde{v}_j - k_\ell) \frac{(\tilde{\Pi}_\ell)^{n-1}}{n}, (\tilde{v}_j - k_{\ell+1})(\tilde{\Pi}_{j-1})^{n-1} \right].$$

Under these conditions,  $\tilde{\pi}_i$  is zero for all  $i \in \{\ell + 1, \dots, j - 1\}$  and  $\tilde{\pi}_\ell$  equals  $(\tilde{\Pi}_j - \tilde{\pi}_j)$  which is, of course,  $\tilde{\Pi}_{j-1}$ .

#### 5.4. Advice concerning Starting Values

If  $\hat{\mathbf{p}}$  is strictly positive for all elements, then whenever a candidate  $\tilde{\pi}_i$  equals zero the Kullback-Leibler distance becomes unbounded, which is a numerical complication. All of the special cases described above are devoted to dealing with such occasions and may seem like a lot of work. Certainly, these cases are tedious to code. However, a good constrained hill-climbing algorithm, like the one coded in SNOPT, can often avoid suboptimal candidates that yield inadmissible  $\tilde{\boldsymbol{\pi}}$ s, provided “good” starting values are used as inputs. How can one provide “good” starting values? One strategy is to find the centre of the simplex  $\mathcal{S}^K$ , which we know is BNC, and then to calculate the  $\dot{\mathbf{v}}$  consistent with it. Use this  $\dot{\mathbf{v}}$  as the starting point for the numerical search.

To calculate the  $\dot{\mathbf{v}}$  consistent with  $\dot{\boldsymbol{\pi}}$  equal  $([1/(K+1)], \dots, [1/(K+1)])$  or a  $\dot{\mathbf{\Pi}}$  of  $([1/(K+1)], [2/(K+1)], \dots, 1)$ , do the following: first, calculate

$$\dot{\Gamma}_i = \frac{(\dot{\Pi}_i)^n - (\dot{\Pi}_{i-1})^n}{n(\dot{\Pi}_i - \dot{\Pi}_{i-1})} \quad i = K, K-1, \dots, 1,$$

and then solve

$$\dot{v}_i = \frac{k_{i+1}\dot{\Gamma}_{i+1} - k_i\dot{\Gamma}_i}{\dot{\Gamma}_i} \quad i = K, K-1, \dots, 1,$$

with  $\dot{v}_0$  being  $k_1$ .

### 5.5. Test Statistic

Given what has come before, performing a test of BNC behaviour is somewhat anti-climactic, at least computationally. Once one has the unconstrained logarithm of the likelihood function  $\mathcal{L}(\hat{\mathbf{p}})$  as well as the largest constrained logarithm of the likelihood function  $\mathcal{L}(\tilde{\boldsymbol{\pi}}^*)$  where  $\tilde{\boldsymbol{\pi}}^* \in \mathcal{C}$ , we propose using the likelihood-ratio test statistic

$$2[\mathcal{L}(\hat{\mathbf{p}}) - \mathcal{L}(\tilde{\boldsymbol{\pi}}^*)] \stackrel{d}{\rightarrow} \chi^2(K). \quad (5.7)$$

to decide whether the constraint binds in a “significant” way. Here we introduce the shorthand notation that  $\tilde{\boldsymbol{\pi}}^*$  equals  $\boldsymbol{\pi}(\tilde{\mathbf{v}}^*)$  where  $\tilde{\mathbf{v}}^*$  solves (5.4). Obviously, the p-values for a  $\chi^2(K)$  can be found easily in a statistics textbook.

### 5.6. Practical Considerations

At least two practical considerations may make implementing the test statistic in (5.7) impossible. First, some of the  $\hat{p}_i$ s may well equal zero, which then makes calculating the Kullback-Leibler distance impossible; one cannot calculate the likelihood-ratio test statistic either. What to do?

In such cases, one might want to use the Euclidean (or Mahalanobis) distance to find  $\tilde{\boldsymbol{\pi}}$ , the closest point in  $\mathcal{C}$ . Given this point, one could then use a test statistic like

$$(\hat{\mathbf{p}}_{-0} - \tilde{\boldsymbol{\pi}}_{-0})^\top [\boldsymbol{\Sigma}(\tilde{\boldsymbol{\pi}}_{-0})]^{-1} (\hat{\mathbf{p}}_{-0} - \tilde{\boldsymbol{\pi}}_{-0}) \quad (5.8)$$

where  $\boldsymbol{\Sigma}(\tilde{\boldsymbol{\pi}}_{-0})$  denotes (4.2) evaluated at  $\tilde{\boldsymbol{\pi}}$  which equals  $\boldsymbol{\pi}(\tilde{\mathbf{v}})$ . Here  $\tilde{\mathbf{v}}$  solves (5.4) with either the Euclidean or the Mahalanobis distance replacing the Kullback-Leibler distance. Under the null hypothesis of BNC behaviour, this statistic is also distributed asymptotically  $\chi^2(K)$ .<sup>6</sup>

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<sup>6</sup> Of course, this strategy does not help if one of the elements of  $\tilde{\boldsymbol{\pi}}$  is exactly equal to zero, since then the inverse of the matrix  $\boldsymbol{\Sigma}(\tilde{\boldsymbol{\pi}}_{-0})$  will not exist. In practice, we have as yet to encounter such an outcome.



Second, the transformation used to create the compact and convex set  $\mathcal{V}(\mathcal{K})$  may introduce local optima in the objective function, regardless of which distance one uses, be it Euclidean, Mahalanobis, or Kulback-Leibler. One commonly-suggested way to check for the presence of these local optima is to try different starting values and then to check to see what happens to the solution. Obviously, this strategy can provide evidence of multiple local optima, but the absence of such evidence clearly does not guarantee that a particular optimum is the global one.

The possibility of local optima is troublesome. For it implies that the likelihood-ratio test [or any other test based on a distance, such as the Wald-type test proposed above (5.8)] will be inconsistent. To wit, its actual size will be less than its nominal size: the test will tend to under-reject the null hypothesis. The reader should note that this is not a property of our transformation of the problem to the  $\mathbf{v}$  space: it would exist in the original specification too.

## 6. Incomplete Inference concerning $\mathbf{F}(\mathbf{v})$

One way to view auction theory is as a specific application of the theory of optimal mechanism design. Over the past four decades, economists have made considerable progress in understanding factors influencing prices realized from goods sold at auction. For example, they have found that the seller’s expected revenue depends on the type of auction employed, the rules that govern bidding, the number of potential bidders, the information available to potential bidders, and the attitudes of bidders toward risk. From a policy-maker’s perspective, however, one of the most important problems involves choosing the selling mechanism that obtains the most profit for the seller. To a large extent, the structure of the optimal selling mechanism depends on the informational environment. Within the IPVP, which was first developed by Vickrey (1961), it is known that, under risk neutrality, four quite different auction formats — the oral, ascending-price (English); the first-price, sealed-bid; the second-price, sealed-bid (Vickrey); and the oral, descending-price (Dutch) auctions — garner the same expected revenue for the seller. This result, which is known as the “Revenue Equivalence Proposition” (REP), is of considerable interest to both economists and policy makers. Given the REP, one question that arises naturally is can one still improve on the structure of the four auction formats? Within a continuous-variation model, Riley and Samuelson (1981) have shown that devising a selling mechanism which maximizes the seller’s expected gain involves choosing an optimal reserve price  $\rho^*$ . Specifically,  $\rho^*$  should be determined according to the following equation:

$$\rho^* = v^S + \frac{[1 - F(\rho^*)]}{f(\rho^*)} \quad (6.1)$$

where  $v^S$  is the seller’s valuation for the object for sale.

The literature concerned with mechanism design has often been criticized as lacking practical value because the optimal selling mechanism depends on random variables whose distributions are typically unknown to the designer. In the past, because the distributions of valuations have been unknown, calculating the optimal reserve price, the optimal selling mechanism, for a real-world auction was impossible. At auctions within the IPVP and with continuous-variation, the equilibrium bidding strategies of potential bidders are increasing functions of their valuations. For example, at first-price, sealed-bid auctions the Bayes-Nash equilibrium bidding strategy is given by (3.4). Thus, in principle, it is possible to estimate the underlying probability law of valuations using the empirical distribution of bids from a cross-section of auctions, and then to use this estimate to estimate the optimal reserve price; Paarsch (1997) provides an example of this with application to English auctions of British Columbian timber.

When the bid data are not continuously recorded, Haile and Tamer (2003) point out, using data from English auctions, that it is impossible to identify the distribution of latent types; one can only estimate bounds using methods developed by Horowitz and Manski (1995). The reason why this obtains is that the statistical model is one of “incomplete inference.” Essentially, a number of different valuations yield the *same* action, preventing the econometrician from inverting the action to get a bidder’s type.

Obviously, this is a problem in our model too. What to do? An alternative, which one might think a useful beginning, would be to assume that  $F(v)$  comes from a parametric family of distributions known up to an unknown  $(p \times 1)$  vector  $\boldsymbol{\theta}$ . Denoting this dependence by  $F(v; \boldsymbol{\theta})$ , one can then calculate  $\pi_i$  as a function of  $\boldsymbol{\theta}$  according to

$$\pi_0(\boldsymbol{\theta}) = F(k_1; \boldsymbol{\theta})$$

and

$$\pi_i(\boldsymbol{\theta}) = [F(v_i; \boldsymbol{\theta}) - F(v_{i-1}; \boldsymbol{\theta})] \quad i = 1, \dots, K - 1$$

with

$$\pi_K(\boldsymbol{\theta}) = [1 - \sum_{i=0}^{K-1} \pi_i(\boldsymbol{\theta})].$$

Now, the logarithm of the likelihood function, ignoring the constant, becomes

$$\sum_{i=0}^K m_i \log[\pi_i(\boldsymbol{\theta})] = m_0 \log[F(k_1; \boldsymbol{\theta})] + \sum_{i=1}^{K-1} m_i \log[F(v_i; \boldsymbol{\theta}) - F(v_{i-1}; \boldsymbol{\theta})] + m_K \log[1 - F(v_K; \boldsymbol{\theta})]$$

which has  $(p + K)$  parameters,  $(\boldsymbol{\theta}, v_1, v_2, \dots, v_K)$ , but only  $K$  pieces of independent information, the  $\mathbf{m}$ . Of course, the  $v_i$ s depend on the  $\boldsymbol{\theta}$ , via equilibrium purposeful behaviour.

Assuming a particular family of  $F(v)$  and then estimating the parameters of this family imposed considerable structure on the DGP. What about non-parametric analysis? Clearly,  $\hat{\mathbf{p}}$  is the non-parametric maximum-likelihood estimator of  $\boldsymbol{\pi}$ , but  $\hat{\mathbf{p}}$  does not contain enough information to estimate  $F(v)$ . As in Haile and Tamer (2003), one can only bound it.

## 7. Experimental Evidence

In order to demonstrate the utility of the approach proposed above, we have conducted a series of experiments. These experiments were conducted in the Laboratoires universitaires Bell, Laboratoire en commerce électronique et économie expérimentale located at CIRANO (Centre interuniversitaire de recherche en analyse des organisations) in Montréal using undergraduate students from a pool of subjects drawn from five universities in Montréal, Québec, Canada, including Concordia University, HEC (École des Hautes Études Commerciales) Montréal, McGill University, Université de Montréal, and UQAM (Université du Québec à Montréal).

### 7.1. Experimental Design

The auctions were conducted electronically using either three or five potential bidders; *i.e.*,  $n$  of three or  $n$  of five. In all experiments, the valuation distribution had support on  $[0, 100]$ . No reserve price existed; all potential bidders were expected to tender some bid. A potential bidder who submitted a bid of 0 still had a chance, albeit a small one, of winning and earning some profit. The bid increment was 10, so  $k_1$  was 0,  $k_2$  was 10,  $k_3$  20, and so forth, making  $\mathcal{K} = \{0, 10, 20, 30, \dots, 90\}$ , with  $k_{K+1}$  being 100.

The subjects were first introduced to the experiment. The subjects were then also offered a bonus which was either a sure \$5 or a lottery of (\$0, \$11) with probabilities (0.5, 0.5).<sup>7</sup> The subjects were initially allowed five practice rounds during which their valuations were drawn from a uniform distribution on  $[0, 100]$ . We discarded these data. Subsequently, we conducted a series of experiments in which the shape of the distribution varied. The distributions of valuations have the following formulae for the probability density functions:

$$f(v) = \begin{cases} \frac{(100-v)}{5,000} & \text{Case a)} \\ \frac{v}{2,500} & v \in [0, 50] \\ \frac{(100-v)}{2,500} & v \in [50, 100] \\ \frac{v}{5,000} & \text{Case c).} \end{cases}$$

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<sup>7</sup> We used this information to get a notion of whether subjects were risk averse or not.

In tables 1–6, we present the theoretical benchmarks for the three types of distributions with three and five players.

Three sessions of the experiment were held over two days, 1 and 2 May 2003. In total, 44 subjects participated: 16 in the first two sessions and 12 in the last session. In each of the three sessions, the subjects were divided into two groups. Subjects were then matched within their own group, having two or four opponents. We recorded data concerning 254 auctions when  $n$  was three and 150 when  $n$  was five, a total of 1,512 observations.

## 7.2. Sample Summary Statistics

In tables 7 and 8, we present the summary statistics for the  $\hat{\mathbf{p}}$ s from our experiments, for  $n$ s of three and five, separately. The columns of the two tables have the realizations from the experiments as well as the theoretical  $\pi_i^0$ s. Listed below each column of realized bid proportions was the total number of observations.

The first thing to note is that the support of the observed bids contains that of the theoretical DGP. To wit, some of the participants bid more than our theoretical model, assuming risk neutrality, would predict. Of course, in many cases, it is only about five percent of the data, but in some (*e.g.*, when  $n$  is three) it can be as high as twenty-five percent of the observations.

It is also interesting to note that, were one to ignore information concerning the valuation distribution generating the experimental data, one could not reject that these data were BNC. Thus, implementing the test for BNC behaviour using equation (4.1) would not result in a rejection of the theory. In the language and notation of Guerre *et al.* (2000), the estimated  $\xi(v)$  function would be monotonic. In tables 9 and 10, we tabulated the theoretical and estimated  $\Gamma_i$ s and  $v_i$ s for the three different distributions a), b), and c) when  $n$  is both three and five, separately. By and large, the experimental data do quite well up to bids of fifty or sixty, but fail to predict well in the tails. For small bids, the  $v_i$ s are estimated quite precisely, but at the endpoints these estimates are quite imprecise.

In tables 11 and 12, we tabulated the  $\hat{\mathbf{p}}$ s for bidders who chose the safe bet over the risky bet when  $n$  was three and five. We call those bidders who chose the safe bet “risk averse” bidders or “Averse” in the tables and those bidders who chose the risky bet “risk neutral” bidders or “Neutral” in the tables. Notice that bidding behaviour is different between these two subgroups. Now, within this experimental framework, two explanations for these behavioural differences exist: first, differences in preferences across decision makers and, second, non-rational behavioural. We then sought to modify our theoretical framework to accommodate these two, potentially-different explanations.

## 8. Asymmetries

In this section, we extend our theoretical model to accommodate asymmetric bidders. We assume that bidders have different types of cumulative distribution functions, indexed by  $g$ . A bidder of type  $g$  gets an independent draw from cumulative distribution function  $F_g(\cdot)$ , while a bidder of type  $h$  gets his independent draw from  $F_h(\cdot)$ . Of course, a bidder of type  $g$  could have the same cumulative distribution function as that for type  $h$ , but different preferences. Different preference functions would manifest themselves in the data as apparent differences in valuation distributions.

We assume that a bidder's type  $g$  is observable by the econometrician, at least *ex post*. We seek to infer from the data how the observable type  $g$  affects bidding behaviour. Under the assumption of risk neutrality, two different assumptions can be made about  $g$ : first, type  $g$  is private information to the bidders and, second, type  $g$  is common knowledge. In the first case, although bidders differ, they are perceived to be *ex ante* symmetric. Obviously, this case is easy to analyse. We do so in subsection 8.1. The second case, when bidders expect others to bid differently from them, is difficult to analyse in general, particularly with continuous variation in bids, but tractable under discrete bidding. We do this in subsection 8.2. In section 9, we consider the case of risk aversion among bidders.

### 8.1. Unknown Types with Symmetric Distributions

We assume here that bidders are of different types  $g$ , but that these types are private information. Denote by  $\omega_g$  the probability of being type  $g$  where  $g = 1, \dots, G$  different types. Suppose that the probability of being a particular type is the same across bidders. Denote by  $\pi_i^g$  the probability that a bidder of type  $g$  submits a bid  $k_i$ . Under these assumptions, the average probability that a given bidder submits  $k_i$  is

$$\pi_i = \sum_{g=1}^G \omega_g \pi_i^g.$$

Using this, one can construct, as before, the vector  $\mathbf{\Gamma}$ . Note that the *ex ante* symmetry implies that the  $\Gamma_i$ s are independent of types. The remainder of the analysis discussed above carries through, particularly that pertaining to Theorem 1.

### 8.2. Bidders with Known Types

When bidders' types are known, the problem become difficult. The main difficulty obtains when calculating the  $\Gamma_i$ s, which will now vary across types. In the continuous-variation world, this computational difficulty manifests itself in the form a system of ordinary differential equations instead of just one, (3.3), which need to be solved simultaneously.

To begin, let  $\pi_i^g$  denote the probability that a bidder of type  $g$  submits a bid of  $k_i$ . We assume that all of the  $\pi_i^g$ s are common knowledge and that, potentially, they differ from one another. Under these assumptions, one can calculate, the probabilities of winning given bid  $k_i$ . Of course, these will vary from one bidder to the next depending on a bidder's type  $g$ . We denote by  $\Gamma_i^g$  the probability that bidder of type  $g$  wins having submitted bid  $k_i$ . Introducing  $\mathcal{N}$  as the set of all  $n$  bidders and  $\mathcal{T}$  as a coalition contained in  $\mathcal{N}$ , for a bidder of type  $g$ , one needs to calculate

$$\Gamma_i^g = \sum_{\mathcal{T} \subseteq \mathcal{N} \setminus g} \left[ \frac{1}{|\mathcal{T}| + 1} \prod_{j \notin \mathcal{T}} \left( \sum_{\ell=0}^{i-1} \pi_\ell^j \right) \prod_{j \in \mathcal{T}} \pi_i^j \right].$$

Numerically, this involves calculating the sum over all  $2^{|\mathcal{N}|-1}$  possible coalitions  $\mathcal{T}$  which are a subset of  $\mathcal{N}$ , having removed all of those who are of type  $g$ . Once the  $\Gamma_i^g$ s have been calculated, one must proceed individually through each type  $g$  to verify whether the analog to (3.1) is valid.

## 9. Risk-Averse Bidders

We can expand our model further by introducing the possibility of risk-averse bidders. Suppose that a bidder of type  $g$  faces the probability of winning  $\Gamma_i^g$  when he submits  $k_i$  and has von Neumann-Morgenstern utility function  $U_g(\cdot)$ . More precisely, suppose that a bidder of type  $g$  having valuation  $v$  solves the following optimization problem:

$$\max_{\langle k_i \in \mathcal{K} \rangle} U_g(v - k_t) \Gamma_t^g.$$

Given some utility function  $U_g(\cdot)$  as well as  $\Gamma^g$ , one can calculate the thresholds  $\underline{v}_i^g$  and  $\bar{v}_i^g$  in a similar fashion. If  $\pi_i^g$  is strictly positive then, as before, one must necessarily have

$$\underline{v}_i^g \leq \bar{v}_i^g$$

with

$$\pi_i^g = [F_g(\bar{v}_i^g) - F_g(\underline{v}_i^g)].$$

Note that, when the utility function is given, as before, one can identify the cumulative distribution function of valuations as some points. Similarly, when the cumulative distribution functions  $F_g(\cdot)$ s are known, one can, from the  $\pi_i^g$ s, calculate the threshold values  $\underline{v}_i^g$  and  $\bar{v}_i^g$  for all  $g$ s and  $i$ s. In turn, this allows one to identify some restrictions on the  $U_g(\cdot)$ s. Note, however, that one cannot identify both the  $U_g(\cdot)$ s and the  $F_g(\cdot)$ s. In future research, we intend to apply the above results to our experimental data.

## 10. Summary and Conclusions

Within the independent private-values paradigm, we have derived the equilibrium implications of purposeful bidding behaviour at single-unit, first-price, sealed-bid auctions when discrete increments are imposed on bidding. Subsequently, we have developed an empirical framework within which to examine why observed data from auctions, either experimental or field, deviates from theoretical predictions. While according with many real-world auctions, the presence of discrete bid increments also simplifies computation when estimating and testing auction models. Our approach fits within a model of incomplete inference and allows us to test sequentially for symmetric, equilibrium purposeful behaviour as well as asymmetric, equilibrium purposeful behaviour. We have demonstrated the utility of the approach by applying it to data from laboratory experiments.

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**Table 1**  
**Case a) with n = 3**

$i$	$k_i$	$v_i$	$F(v_i)$	$\Gamma_i$	$v_i^*$	$\pi_i^0$
1	0	11	0.2079	0.0144	11.5307	0.2079
2	10	25	0.4375	0.1085	25.6525	0.2296
3	20	41	0.6519	0.3005	41.8980	0.2144
4	30	59	0.8319	0.5531	59.1918	0.1800
5	40	100	1.0000	0.8413	103.0196	0.1681
6	50	100	1.0000	1.0000		0.0000

**Table 2**  
**Case a) with n = 5**

$i$	$k_i$	$v_i$	$F(v_i)$	$\Gamma_i$	$v_i^*$	$\pi_i^0$
1	0	10	0.1900	0.0003	10.3453	0.1900
2	10	21	0.3759	0.0078	21.7354	0.1859
3	20	34	0.5644	0.0528	34.1202	0.1885
4	30	48	0.7296	0.1810	48.2745	0.1652
5	40	62	0.8556	0.3996	62.1231	0.1260
6	50	89	0.9879	0.7293	89.5511	0.1323
7	60	100	1.0000	0.9761	478.2535	0.0121
8	70	100	1.0000	1.0000		0.0000

**Table 3**  
Case b) with  $n = 3$

$i$	$k_i$	$v_i$	$F(v_i)$	$\Gamma_i$	$v_i^*$	$\pi_i^0$
1	0	10	0.0200	0.0001	10.4191	0.0200
2	10	21	0.0882	0.0033	21.5420	0.0682
3	20	33	0.2178	0.0248	33.3042	0.1296
4	30	45	0.4050	0.0999	45.4951	0.1872
5	40	58	0.6472	0.2817	58.3196	0.2422
6	50	80	0.9200	0.6202	80.5441	0.2728
7	60	100	1.0000	0.9221	188.4247	0.0800
8	70	100	1.0000	1.0000		0.0000

**Table 4**  
Case b) with  $n = 5$

$i$	$k_i$	$v_i$	$F(v_i)$	$\Gamma_i$	$v_i^*$	$\pi_i^0$
1	0	10	0.0200	0.0000	10.0294	0.0200
2	10	20	0.0800	0.0000	20.3027	0.0600
3	20	30	0.1800	0.0004	30.7560	0.1000
4	30	41	0.3362	0.0053	41.4846	0.1562
5	40	52	0.5392	0.0407	52.8157	0.2030
6	50	65	0.7550	0.1851	65.2114	0.2158
7	60	84	0.9488	0.5403	84.9110	0.1938
8	70	100	1.0000	0.9027	172.7855	0.0512
9	80	100	1.0000	1.0000		0.0000

**Table 5**  
Case c) with  $n = 3$

$i$	$k_i$	$v_i$	$F(v_i)$	$\Gamma_i$	$v_i^*$	$\pi_i^0$
1	0	10	0.0100	0.0000	10.4191	0.0100
2	10	21	0.0441	0.0008	21.5420	0.0341
3	20	33	0.1089	0.0062	33.3042	0.0648
4	30	45	0.2025	0.0250	45.8558	0.0936
5	40	56	0.3136	0.0676	56.5928	0.1111
6	50	71	0.5041	0.1702	71.4043	0.1905
7	60	79	0.6241	0.3194	79.0795	0.1200
8	70	100	1.0000	0.6712	100.4137	0.3759
9	80	100	1.0000	1.0000		0.0000

**Table 6**  
Case c) with  $n = 5$

$i$	$k_i$	$v_i$	$F(v_i)$	$\Gamma_i$	$v_i^*$	$\pi_i^0$
1	0	10	0.0100	0.0000	10.0294	0.0100
2	10	20	0.0400	0.0000	20.3027	0.0300
3	20	30	0.0900	0.0000	30.7600	0.0500
4	30	41	0.1681	0.0003	41.4702	0.0781
5	40	52	0.2704	0.0026	52.3909	0.1023
6	50	63	0.3969	0.0133	63.1022	0.1265
7	60	75	0.5625	0.0561	75.1189	0.1656
8	70	84	0.7056	0.1657	84.1982	0.1431
9	80	100	1.0000	0.5605	102.7546	0.2944
10	90	100	1.0000	1.0000		0.0000

**Table 7**  
**Summary Statistics for n = 3**

$i$	$k_i$	$\hat{p}_i^a$	$\pi_i^{0,a}$	$\hat{p}_i^b$	$\pi_i^{0,b}$	$\hat{p}_i^c$	$\pi_i^{0,c}$
1	0	0.2208	0.2079	0.0296	0.0200	0.0238	0.0100
2	10	0.2250	0.2296	0.0926	0.0682	0.0437	0.0341
3	20	0.1375	0.2144	0.1519	0.1296	0.0556	0.0648
4	30	0.1583	0.1800	0.1370	0.1872	0.0595	0.0936
5	40	0.1208	0.1681	0.2148	0.2422	0.0754	0.1111
6	50	0.0958	0.0000	0.1852	0.2728	0.1508	0.1905
7	60	0.0333		0.1111	0.0800	0.1349	0.1200
8	70	0.0083		0.0519	0.0000	0.1944	0.3759
9	80			0.0148		0.1944	0.0000
10	90			0.0111		0.0675	
$nT$		240		270		252	

**Table 8**  
**Summary Statistics for n = 5**

$i$	$k_i$	$\hat{p}_i^a$	$\pi_i^{0,a}$	$\hat{p}_i^b$	$\pi_i^{0,b}$	$\hat{p}_i^c$	$\pi_i^{0,c}$
1	0	0.2200	0.1900	0.0360	0.0200	0.0480	0.0100
2	10	0.1880	0.1859	0.0480	0.0600	0.0440	0.0300
3	20	0.1640	0.1885	0.1400	0.1000	0.0600	0.0500
4	30	0.1200	0.1652	0.1200	0.1562	0.0840	0.0781
5	40	0.1280	0.1260	0.1680	0.2030	0.0520	0.1023
6	50	0.1040	0.1323	0.2040	0.2158	0.0920	0.1265
7	60	0.0440	0.0121	0.1200	0.1938	0.1560	0.1656
8	70	0.0240	0.0000	0.1280	0.0512	0.1760	0.1431
9	80	0.0040		0.0320	0.0000	0.2240	0.2944
10	90	0.0040		0.0040		0.0640	0.0000
$nT$		250		250		250	

**Table 9**  
Cases a), b), and c) with  $n = 3$

$i$	$k_i$	$\Gamma_i^a$	$\hat{\Gamma}_i^a$	$v_i^{*,a}$	$\hat{v}_i^{*,a}$	$\Gamma_i^b$	$\hat{\Gamma}_i^b$	$v_i^{*,b}$	$\hat{v}_i^{*,b}$	$\Gamma_i^c$	$\hat{\Gamma}_i^c$	$v_i^{*,c}$	$\hat{v}_i^{*,c}$
1	0	0.014	0.016	11.531	11.641	0.000	0.000	10.419	10.472	0.000	0.000	10.419	10.919
2	10	0.109	0.115	25.653	27.635	0.003	0.007	21.542	21.866	0.001	0.002	21.542	23.161
3	20	0.301	0.266	41.898	45.273	0.025	0.041	33.304	35.297	0.006	0.009	33.304	36.486
4	30	0.553	0.441	59.192	61.687	0.100	0.119	45.495	47.736	0.025	0.024	45.856	49.315
5	40	0.841	0.644	103.020	84.822	0.282	0.273	58.320	61.066	0.068	0.049	56.593	57.673
6	50	1.000	0.829		128.515	0.620	0.519	80.544	82.278	0.170	0.113	71.404	69.823
7	60		0.950		302.307	0.922	0.752	188.425	121.129	0.319	0.229	79.080	82.319
8	70		0.991			1.000	0.899		220.179	0.671	0.415	100.414	94.436
9	80						0.963		466.599	1.000	0.702		120.081
10	90						0.989				0.935		

**Table 10**  
Cases a), b), and c) with  $n = 5$

$i$	$k_i$	$\Gamma_i^a$	$\hat{\Gamma}_i^a$	$v_i^{*,a}$	$\hat{v}_i^{*,a}$	$\Gamma_i^b$	$\hat{\Gamma}_i^b$	$v_i^{*,b}$	$\hat{v}_i^{*,b}$	$\Gamma_i^c$	$\hat{\Gamma}_i^c$	$v_i^{*,c}$	$\hat{v}_i^{*,c}$
1	0	0.000	0.001	10.345	10.426	0.000	0.000	10.029	10.026	0.000	0.000	10.029	10.102
2	10	0.008	0.012	21.735	22.323	0.000	0.001	20.303	20.303	0.000	0.000	20.303	20.707
3	20	0.053	0.061	34.120	35.997	0.000	0.002	30.757	31.554	0.000	0.000	30.760	31.952
4	30	0.181	0.162	48.275	49.615	0.005	0.015	41.485	42.339	0.000	0.001	41.470	43.076
5	40	0.400	0.331	62.123	63.203	0.041	0.079	52.816	53.988	0.003	0.003	52.391	52.249
6	50	0.729	0.582	89.551	86.535	0.185	0.275	65.211	69.386	0.013	0.014	63.102	63.424
7	60	0.976	0.802	478.254	136.312	0.540	0.568	84.911	93.636	0.056	0.054	75.119	74.358
8	70	1.000	0.923		252.205	0.903	0.809	172.786	148.163	0.166	0.177	84.198	85.461
9	80		0.976		707.515	1.000	0.928		275.957	0.562	0.501	102.755	103.387
10	90		0.992				0.978			1.000	0.876		

**Table 11**  
 **$\hat{p}$ : Risk-Neutral and Risk-Averse Bidders,  $n = 3$**   
**Cases a), b), and c)**

Type/ $k_i$	0	10	20	30	40	50	60	70	80	90
Neutral	0.2372	0.2372	0.1090	0.1731	0.1346	0.0897	0.0192	0.0000	0.0000	0.0000
Averse	0.1905	0.2024	0.1905	0.1310	0.0952	0.1071	0.0595	0.0238	0.0000	0.0000
Neutral	0.0333	0.1000	0.1556	0.1444	0.2000	0.1611	0.1056	0.0667	0.0222	0.0111
Averse	0.0222	0.0778	0.1444	0.1222	0.2444	0.2333	0.1222	0.0222	0.0000	0.0111
Neutral	0.0120	0.0542	0.0663	0.0723	0.0783	0.1566	0.1325	0.1747	0.1867	0.0663
Averse	0.0465	0.0233	0.0349	0.0349	0.0698	0.1395	0.1395	0.2326	0.2093	0.0698

**Table 12**  
 **$\hat{p}$ : Risk-Neutral and Risk-Averse Bidders,  $n = 5$**   
**Cases a), b), and c)**

Type/ $k_i$	0	10	20	30	40	50	60	70	80	90
Neutral	0.2171	0.1860	0.1550	0.1163	0.1085	0.1085	0.0543	0.0388	0.0078	0.0078
Averse	0.2231	0.1901	0.1736	0.1240	0.1488	0.0992	0.0331	0.0083	0.0000	0.0000
Neutral	0.0400	0.0514	0.1143	0.1143	0.2000	0.1714	0.1429	0.1257	0.0343	0.0057
Averse	0.0267	0.0400	0.2000	0.1333	0.0933	0.2800	0.0667	0.1333	0.0267	0.0000
Neutral	0.0682	0.0606	0.0682	0.0606	0.0455	0.0682	0.1667	0.1439	0.2348	0.0833
Averse	0.0254	0.0254	0.0508	0.1102	0.0593	0.1186	0.1441	0.2119	0.2119	0.0424