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Option Valuation with Conditional Heteroskedasticity and Non-Normality[*]

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Résumé

Nous présentons les résultats d'une étude portant sur l'évaluation de créances éventuelles de style européen pour une grande variété de caractéristiques liées au rendement des actifs sous-jacents. Les résultats de notre évaluation proposent en temps discret une formule état-espace infinie, à partir du principe de non-arbitrage et d'une mesure de martingale équivalente. Notre approche permet de tenir compte de formes générales d'hétéroscédasticité dans les rendements et d'obtenir, dans des cas spéciaux, des résultats d'évaluation liés aux processus homoscédastiques. Elle permet aussi de considérer les innovations conditionnellement non normales en matière de rendement, ce qui représente un facteur critique, compte tenu du fait que l'hétéroscédasticité ne permet pas, à elle seule, de saisir pleinement le caractère ironique de l'option. Nous analysons une catégorie de mesures de martingale équivalentes dont la dynamique du rendement risque-neutre obtenu est de la même famille de distribution que la dynamique du rendement physique. Dans ce cas, notre cadre d'étude soutient les résultats d'évaluation obtenus par Duan (1995) et par Heston et Nandi (2000) et tient compte du coût du risque variant dans le temps et des innovations non normales. Nous étendons ces résultats aux mesures de martingale équivalentes plus générales et aux modèles de volatilité stochastique en temps discret et analysons aussi la relation entre nos résultats et ceux obtenus dans le cas des modèles en temps continu.

> **Mots clés** : GARCH (hétéroscédasticité conditionnelle autorégressive généralisée), évaluation du risque neutre, absence d'arbitrage, innovations non normales

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Abstract

We provide results for the valuation of European style contingent claims for a large class of specifications of the underlying asset returns. Our valuation results obtain in a discrete time, infinite state-space setup using the no-arbitrage principle and an equivalent martingale measure. Our approach allows for general forms of heteroskedasticity in returns, and valuation results for homoskedastic processes can be obtained as a special case. It also allows for conditional non-normal return innovations, which is critically important because heteroskedasticity alone does not suffice to capture the option smirk. We analyze a class of equivalent martingale measures for which the resulting risk-neutral return dynamics are from the same family of distributions as the physical return dynamics. In this case, our framework nests the valuation results obtained by Duan (1995) and Heston and Nandi (2000) by allowing for a time-varying price of risk and non-normal innovations. We provide extensions of these results to more general equivalent martingale measures and to discrete time stochastic volatility models, and we analyze the relation between our results and those obtained for continuous time models.

1 Introduction

A contingent claim is a security whose payoff depends upon the value of another underlying security. A valuation relationship is an expression that relates the value of the contingent claim to the value of the underlying security and other variables. The most popular approach for valuing contingent claims is the use of a Risk Neutral Valuation Relationship (RNVR).

Most of the literature on contingent claims and most of the applications of the RNVR have been cast in continuous time. While the continuous-time approach offers many advantages, the valuation of contingent claims in discrete time is also of substantial interest. For example, when hedging option positions, rebalancing decisions must be made in discrete time, and in the case of American and exotic options, early exercise decisions must be made in discrete time. However, by far the most important advantage of working in discrete time is econometric convenience. It is difficult to estimate continuous-time processes, because of the complexity of the resulting filtering problem for processes that adequately capture stylized facts, such as Heston's (1993a) stochastic volatility model. In contrast, for many of the models we study in this paper, the resulting filtering problem is extremely simple.

Because of the econometric convenience, most of the stylized facts characterizing underlying securities have been studied in discrete-time models. One very important feature of returns is conditional heteroskedasticity, which can be addressed in the GARCH framework of Engle (1982) and Bollerslev (1986).[1] Presumably, because of this evidence, most of the recent empirical work on discrete-time option valuation has also focused on GARCH processes.[2] The GARCH model amounts to an infinite state space setup, with the innovations for underlying asset returns described by continuous distributions. In this case the market is incomplete, and it is in general not possible to construct a portfolio containing combinations of the contingent claim and the underlying asset that make the resulting portfolio riskless.[3]

To obtain a RNVR, the GARCH option valuation literature builds on the approach of Rubinstein (1976) and Brennan (1979), who demonstrate how to obtain RNVRs for lognormal and normal returns in the case of constant mean return and volatility, by specifying a representative agent economy and characterizing sufficient conditions on preferences. For a given dynamic of

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[1]See for example French, Schwert and Stambaugh (1987) and Schwert (1989) for early studies on stock returns. The literature is far too voluminous to cite all relevant papers here. See Bollerslev, Chou and Kroner (1992) and Diebold and Lopez (1995) for reviews on GARCH modeling.

[2]See Bollerslev and Mikkelsen (1996), Satchell and Timmermann (1996), Garcia and Renault (1998), Heston and Nandi (2000), Christoffersen and Jacobs (2004), Christoffersen, Heston and Jacobs (2006), and Barone-Adesi, Engle and Mancini (2008) for applications to option valuation.

[3]In a discrete time finite state space setting, Harrison and Pliska (1981) provide the mathematical framework to obtain the existence of the risk neutral probability measure, to demonstrate uniqueness in the case of complete markets, and to get a RNVR for any contingent claim. See also Harrison and Kreps (1979), Cox, Ross and Rubinstein (1979) and Cox and Ross (1976) for discrete-time finite state-space approaches.

the underlying security, specific assumptions have to be made on preferences in order to obtain a risk neutralization result.[4] The first order condition resulting from this economy yields an Euler equation that can be used to price any asset. For lognormal stock returns and a conditionally heteroskedastic (GARCH) volatility dynamic, the standard result is the one in Duan (1995). Duan's result relies on the existence of a representative agent with constant relative risk aversion or constant absolute risk aversion.[5]

Because it is difficult to characterize the general equilibrium setup underlying a RNVR, very few valuation results are currently available for heteroskedastic processes with non-normal innovations.[6] In this paper, we argue that it is possible to investigate option valuation for a large class of conditionally non-normal heteroskedastic processes, provided that the conditional moment generating function (MGF) exists. It is also possible to accommodate a large class of time-varying risk premia. Our framework differs from the approach in Brennan (1979) and Duan (1995), and is more intimately related to the approach adopted in continuous-time option valuation: we only use the no-arbitrage assumption and some technical conditions on the investment strategies to show the existence of an RNVR. We demonstrate the existence of an EMM and characterize it, without first making an explicit assumption on the utility function of a representative agent. We then show that the price of the contingent claim defined as the expected value of the discounted payoff at maturity is a no-arbitrage price and characterize the risk-neutral dynamic. We provide results for GARCH processes and for more general discrete-time stochastic volatility models. We also analyze several important limit results for the discrete-time processes we consider, and we discuss the relationships between risk-neutralization in these models and continuous-time stochastic volatility models.

Why are we able to provide more general valuation results than the existing literature? In our opinion, the analysis in Brennan (1979) and Duan (1995) addresses two important questions simultaneously: First, a mostly technical question that characterizes the risk-neutral dynamic and the valuation of options; second, a more economic one that characterizes the equilibrium underlying the valuation procedure. The existing discrete-time literature for the most part has viewed these two questions as inextricably linked, and has therefore largely limited itself to

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[4]Brennan (1979) characterizes the bivariate distribution of returns on aggregate wealth and the underlying asset under which a risk-neutral valuation relationship obtains in the homoskedastic case. Camara (2003) uses this approach to obtain valuation results for transformed normal dynamics of returns and state variables. See also Schroder (2004).

[5]See also Amin and Ng (1993) who study the heteroskedastic case by making an assumption on the bivariate distribution of the stochastic discount factor and the underlying return process.

[6]Duan, Ritchken and Sun (2005) analyze a heteroskedastic model with Poisson-normal innovations and Duan (1999) analyzes a conditionally fat-tailed heteroskedastic model. Christoffersen, Heston and Jacobs (2006) use a heteroskedastic return dynamic with inverse Gaussian innovations. Other studies analyze non-normal innovations. Madan and Seneta (1990) use the symmetric and i.i.d. variance gamma distribution. Heston (1993b) presents results for the gamma distribution and Heston (2004) analyzes a number of infinitely divisible distributions.

(log)normal return processes as well as a few special non-normal cases. Our paper differs in a subtle but important way from most existing studies. We argue that it is possible and desirable to treat these questions one at a time. We do not attempt to characterize the preferences underlying the risk-neutral valuation relationship. Instead, we assume a class of Radon-Nikodym derivatives and search for an EMM within this class. This allows us to provide some general results on the valuation of options under conditionally non-normal asset returns without fully characterizing the economic environment. We also show how the normal model and available conditional non-normal models are special cases of our setup.

The same approach of separating these two questions occurs in the literature on option valuation using continuous-time stochastic volatility models, such as for instance in Heston's (1993a) model. These models yield different equivalent martingale measures for different specifications of the volatility risk premium. For a given specification of the volatility risk premium, one can find an EMM and characterize the risk-neutral dynamic using Girsanov's theorem. To derive this result, and to value options, there is no need to explicitly characterize the utility function underlying the volatility risk premium. The latter task is very interesting in its own right, but differs from characterizing the risk-neutral dynamic and the option value for a given physical return dynamic.[7] The latter is a purely mathematical exercise. The former provides the economic background behind a particular choice of volatility premium, and therefore helps us understand whether a particular choice of functional form for the risk premium, which is often made for convenience, is also reasonable from an economic perspective.

The paper proceeds as follows. In Section 2 we define a class of heteroskedastic stock return processes, and we characterize the condition for an EMM for this class of processes. We then show sufficient conditions for an EMM to exist and we derive the risk neutral distribution of returns. In Section 3 we further discuss the choice of EMM in Section 2, and introduce a more general class of EMMs. Section 4 derives the no-arbitrage option price corresponding to the EMM. Section 5 discusses several special cases of return dynamics that can be analyzed using our approach. Section 6 provides continuous-time limits of a number of important models. Section 7 introduces an extension to discrete-time stochastic volatility models and compares it with the benchmark continuous time model. Section 8 concludes.

2   Conditionally heteroskedastic models

In Section 2.1 we define the stock price process that we use in Sections 2 through 6. This process is able to accommodate the class of ARCH and GARCH processes. In Sections 2.2-2.6, we

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[7]See for instance Heston (1993a) and Bates (1996, 2000) for a discussion.

then analyze the risk-neutralization of this stock price process using a particularly convenient candidate Radon-Nikodym derivative.

We use P to describe the physical distribution of the states of nature. The financial market consists of a zero-coupon risk-free bond index and a stock. The dynamics of the bond are described by the process $\{B_t\}_{t=0}^{T}$ normalized to $B_0 = 1$ and the dynamics of the stock price by $\{S_t\}_{t=0}^{T}$. The information structure is given by the filtration $\mathbb{F} = \{F_t \mid t = 0, ..., T\}$ generated by the stock and the bond process.

2.1   The stock price process

The underlying stock price process is assumed to follow the conditional distribution D under the physical measure P. We write

$$R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = \mu_t - \gamma_t + \varepsilon_t \qquad\qquad \varepsilon_t | F_{t-1} \sim D(0, \sigma_t^2) \qquad (2.1)$$

where S_t is the stock price at time t, and σ_t^2 is the conditional variance of the log return in period t. The mean correction factor, γ_t, is defined from

$$\exp(\gamma_t) \equiv E_{t-1}\left[\exp(\varepsilon_t)\right]$$

and it serves to ensure that the conditional expected gross rate of return, $E_{t-1}\left[S_t / S_{t-1}\right]$, is equal to $\exp(\mu_t)$. More explicitly,

$$
\begin{aligned}
E_{t-1}\left[S_t / S_{t-1}\right] &= E_{t-1}\left[\exp(\mu_t - \gamma_t + \varepsilon_t)\right] = \exp(\mu_t) \\
&\Longleftrightarrow \exp(\gamma_t) = E_{t-1}\left[\exp(\varepsilon_t)\right]
\end{aligned}
$$

Note that our specification (2.1) does not restrict the risk premium in any way, nor does it assume conditional normality.

For now, we follow most of the existing discrete-time empirical finance literature by focusing on conditional means μ_t and conditional variances σ_t^2 that are F_{t-1} measurable. We will relax this assumption in Section 7. We do not constrain the interest rate r_t to be constant. It is instead assumed to be an element of F_{t-1} as well. This setup is able to accommodate the class of ARCH and GARCH processes proposed by Engle (1982) and Bollerslev (1986) and used for option valuation by Amin and Ng (1993), Duan (1995, 1999), and Heston and Nandi (2000). Our results also hold for different types of GARCH specifications, such as the EGARCH model of Nelson (1991) or the specification of Glosten, Jagannathan and Runkle (1993).

In the following, we show that we can find an EMM by defining a probability measure that

makes the discounted security process a martingale. We derive more general results on option valuation for heteroskedastic processes compared to the available literature, because we focus on the narrow question of option valuation while ignoring the economic question regarding the preferences of the representative agent that support this valuation argument in equilibrium.

We use a no-arbitrage argument that is similar to the one used in the continuous-time literature. We first prove the existence of an EMM. Subsequently we demonstrate the existence of a RNVR by demonstrating that the price of the contingent claim, defined as the expected value of the discounted payoff at maturity, is a no-arbitrage price under this EMM.[8] The proof uses an argument similar to the one used in the continuous-time literature, but is arguably more straightforward as it avoids the technical issues involved in the analysis of local and super martingales.

2.2 Specifying an equivalent martingale measure

The objective in this section is to find a measure equivalent to the physical measure P that makes the price of the stock discounted by the riskless asset a martingale. An EMM is defined as long as the Radon-Nikodym derivative is defined. We start by specifying a candidate Radon-Nikodym derivative of a probability measure. We then show that this Radon-Nikodym derivative defines an EMM that makes the discounted stock price process a martingale. This result in turn allows us to obtain the distribution of the stock return under this EMM.

For a given predetermined sequence, $\{\nu_t\}$, we define the following candidate Radon-Nikodym derivative

$$\frac{dQ}{dP}\bigg| F_t = \exp\left(-\sum_{i=1}^{t} \left(\nu_i \varepsilon_i + \Psi_i\left(\nu_i\right)\right)\right) \qquad (2.2)$$

where $\Psi_t\left(u\right)$ is defined as the natural logarithm of the moment generating function

$$E_{t-1}\left[\exp(-u\varepsilon_t)\right] \equiv \exp\left(\Psi_t\left(u\right)\right)$$

Note that we can think of the mean correction factor in (2.1) as $\gamma_t = \Psi_t\left(-1\right)$. Note also that in the normal case we have $\Psi_t\left(u\right) = \frac{1}{2}\sigma_t^2 u^2$ and $\gamma_t = \Psi_t\left(-1\right) = \frac{1}{2}\sigma_t^2$.

We can now show the following lemma

Lemma 1 $\frac{dQ}{dP}\big| F_t$ *is a Radon-Nikodym derivative*

---

[8]Duan (1995) refers to RNVR as Local RNVR in the case of GARCH. The reason for the distinction is that (under normality) the conditional volatility is identical under the two measures only one period ahead. In the remainder of the paper we will drop this distinction for ease of exposition. We emphasize that the result that the conditional volatility differs between the two measures for more than one period ahead is to be expected as volatility is random in this case. This feature is very similar to the continuous time case, which has random volatility for any horizon.

Proof. We need to show that $\frac{dQ}{dP}\big| F_t > 0$ which is immediate. We also need to show that $E_0^P\left[\frac{dQ}{dP}\big| F_t\right] = 1$. We have

$$E_0^P\left[\frac{dQ}{dP}\bigg| F_t\right] = E_0^P\left[\exp\left(-\sum_{i=1}^{t}\left(\nu_i\varepsilon_i + \Psi_i\left(\nu_i\right)\right)\right)\right].$$

Using the law of iterative expectations we can write

$$
\begin{aligned}
E_0^P\left[\frac{dQ}{dP}\bigg| F_t\right] &= E_0^P\left[E_1^P...E_{t-1}^P\exp\left(-\sum_{i=1}^{t}\left(\nu_i\varepsilon_i + \Psi_i\left(\nu_i\right)\right)\right)\right]\\
&= E_0^P\left[E_1^P...E_{t-2}^P\exp\left(-\sum_{i=1}^{t-1}\nu_i\varepsilon_i - \sum_{i=1}^{t}\Psi_i\left(\nu_i\right)\right)E_{t-1}^P\exp\left(-\nu_t\varepsilon_t\right)\right]\\
&= E_0^P\left[E_1^P...E_{t-2}^P\exp\left(-\sum_{i=1}^{t-1}\nu_i\varepsilon_i - \sum_{i=1}^{t}\Psi_i\left(\nu_i\right)\right)\exp\left(\Psi_t\left(\nu_t\right)\right)\right]\\
&= E_0^P\left[E_1^P...E_{t-2}^P\exp\left(-\sum_{i=1}^{t-1}\nu_i\varepsilon_i - \sum_{i=1}^{t-1}\Psi_i\left(\nu_i\right)\right)\right]
\end{aligned}
$$

Iteratively, using this result we get

$$
\begin{aligned}
E_0^P\left[\frac{dQ}{dP}\bigg| F_t\right] &= E_0^P\left[\exp\left(-\nu_1\varepsilon_1 - \Psi_1\left(\nu_1\right)\right)\right]\\
&= \exp\left(-\Psi_1\left(\nu_1\right)\right)\exp\left(\Psi_1\left(\nu_1\right)\right) = 1
\end{aligned}
$$

and the lemma obtains. ∎

We are now ready to show that we can specify an EMM using this Radon-Nikodym derivative.

Proposition 1 *The probability measure Q defined by the Radon-Nikodym derivative in (2.2) is an EMM if and only if*

$$\Psi_t\left(\nu_t - 1\right) - \Psi_t\left(\nu_t\right) - \gamma_t + \phi_t\sigma_t^2 = 0 \tag{2.3}$$

where $\phi_t = \dfrac{\mu_t - r_t}{\sigma_t^2}$.

Proof. We need $E^Q\left[\dfrac{S_t}{B_t}\bigg| F_{t-1}\right] = \dfrac{S_{t-1}}{B_{t-1}}$ or equivalently $E^Q\left[\dfrac{S_t}{S_{t-1}}\bigg/\dfrac{B_t}{B_{t-1}}\bigg| F_{t-1}\right] = 1$. We have

$$E^Q \left[\left. \frac{S_t}{S_{t-1}} \middle/ \frac{B_t}{B_{t-1}} \right| F_{t-1} \right] = E^P \left[\left(\frac{\frac{dQ}{dP} \middle| F_t}{\frac{dQ}{dP} \middle| F_{t-1}} \right) \frac{S_t}{S_{t-1}} \middle/ \frac{B_t}{B_{t-1}} \middle| F_{t-1} \right]$$

$$= E^P \left[\left(\frac{\frac{dQ}{dP} \middle| F_t}{\frac{dQ}{dP} \middle| F_{t-1}} \right) \frac{S_t}{S_{t-1}} \exp(-r_t) \middle| F_{t-1} \right]$$

$$= E^P \left[\exp\left(-\nu_t \varepsilon_t - \Psi_t \left(\nu_t \right) \right) \exp(\mu_t - \gamma_t + \varepsilon_t) \exp(-r_t) \middle| F_{t-1} \right]$$

$$= \exp\left(-\Psi_t \left(\nu_t \right) + \mu_t - r_t - \gamma_t \right) E^P \left[\exp\left(\left(1 - \nu_t \right) \varepsilon_t \right) \middle| F_{t-1} \right]$$

$$= \exp\left(-\Psi_t \left(\nu_t \right) + \mu_t - r_t - \gamma_t + \Psi_t \left(\nu_t - 1 \right) \right)$$

Thus Q is a probability measure that makes the stock discounted by a riskless asset a martingale if and only if

$$\Psi_t \left(\nu_t - 1 \right) - \Psi_t \left(\nu_t \right) - \gamma_t + \phi_t \sigma_t^2 = 0 \tag{2.4}$$

This result implies that we can construct an EMM by choosing the sequence $\{\nu_t\}$ to make (2.4) hold.[9] ■

2.3 Solving for the EMM

In this section we develop various results on the existence of a solution to (2.4), conditional on our assumption regarding the family of Radon-Nikodym derivatives.

Note first that in the conditional normal special case we get the solution to be the well-known price of risk $\nu_t = \phi_t = (\mu_t - r_t)/\sigma_t^2$. Note also that if we additionally specify the conditional mean of the excess return to be affine in σ_t^2, so that $\mu_t = r_t + \lambda \sigma_t^2$, then ν_t is simply a constant λ.

When allowing for conditional non-normal returns, we need to put some structure on $\Psi_t \left(. \right)$ in order to analyze the existence of a solution to (2.4). In Section 5 below we consider some important non-normal special cases where an explicit solution for ν_t can be found. More generally, we provide the following result.

Proposition 2 *If Ψ is strictly convex, twice differentiable, and tends to infinity at the boundaries of its domain (u_1, u_2) where $u_1 + 1 < u_2$, then there exists a solution to equation (2.4). This solution is unique. Note that u_1 and u_2 are not restricted to be finite.*

Proof. See the Appendix. ■

---

[9] See Shiryaev (1999) for an introduction to the conditional use of the Radon-Nikodym derivative.

Proposition 2 provides a set of sufficient, not necessary, conditions for a unique solution to exist. A similar result can be obtained assuming that Ψ is strictly concave. However, the parametric examples we consider below are part of the class of infinitely divisible distributions, thus ensuring that strict convexity holds (Feller, 1968), and therefore the strict convexity assumption in Proposition 2 is more realistic for our purposes.[10] In Section 5 below, we discuss the other conditions in Proposition 2 on a case-by-case basis, and thus verify that overall these conditions are very reasonable.

In the absence of sufficient conditions, an approximate solution to the EMM equation in (2.4) can be obtained from the second-order approximations

$$\Psi_t\left(\nu_t - 1\right) \approx \Psi_t\left(0\right) + \Psi_t'\left(0\right)\left(\nu_t - 1\right) + \tfrac{1}{2}\Psi_t''\left(0\right)\left(\nu_t - 1\right)^2$$
$$\Psi_t\left(\nu_t\right) \approx \Psi_t\left(0\right) + \Psi_t'\left(0\right)\nu_t + \tfrac{1}{2}\Psi_t''\left(0\right)\nu_t^2$$

From the definition of the mean-zero shock ε_t we have that $\Psi_t'\left(0\right) = E_{t-1}\left[\varepsilon_t\right] = 0$, and $\Psi_t''\left(0\right) = Var_{t-1}\left[\varepsilon_t\right] = \sigma_t^2$, so that the approximation along with the EMM condition (2.4) gives us

$$\nu_t \approx \frac{\mu_t - r_t}{\sigma_t^2} + \frac{1}{2} - \frac{\gamma_t}{\sigma_t^2} \tag{2.5}$$

Notice that this approximation is exact in the normal case, where $\gamma_t = \tfrac{1}{2}\sigma_t^2$ and $\nu_t = \left(\mu_t - r_t\right)/\sigma_t^2$. This approximate solution can be used in place of the exact solution, or it can be used as a starting value in a numerical search for the exact ν_t.

Note finally that (2.4) suggests that the problem of finding a solution for ν_t can be circumvented altogether if one is willing to put more structure on the return process in (2.1). Denote the risk premium $\mu_t - r_t$ by π_t. If the return dynamic is specified such that

$$R_t = r_t + \pi_t - \gamma_t + \varepsilon_t \qquad\qquad \varepsilon_t|F_{t-1} \sim D(0, \sigma_t^2)$$

where

$$\pi_t = \Psi_t\left(\nu_t\right) - \Psi_t\left(\nu_t - 1\right) + \gamma_t, \tag{2.6}$$

then the EMM condition in (2.4) is trivially satisfied for any value of ν_t. Thus ν_t can be set to a constant ν, to be estimated as part of the return dynamic. This approach is viable but suffers from the drawback that the return mean dynamic is now motivated by convenience rather than

---

[10]Gourieroux and Monfort (2007) provide similar conditions in a setup with a stochastic discount factor. They do not relate their result to the class of inifinitely divisible distributions.

empirical relevance. Note that in the normal case this approach yields

$$
\begin{aligned}
R_t &= r_t + \Psi_t\left(\nu\right) - \Psi_t\left(\nu - 1\right) + \varepsilon_t && \varepsilon_t|F_{t-1} \sim N(0, \sigma_t^2) \\
&= r_t + \nu\sigma_t^2 - \tfrac{1}{2}\sigma_t^2 + \varepsilon_t
\end{aligned}
$$

which corresponds to an affine risk premium.

2.4   Characterizing the risk-neutral distribution

When pricing options using Monte Carlo simulation, knowing the risk neutral distribution is valuable. In this section, we derive an important result that shows that for the class of models we investigate and using the class of Radon-Nikodym derivatives in (2.2), the risk neutral distribution is from the same family as the original physical distribution.

We first need the following lemma where we recall that $\Psi_t\left(u\right)$ denotes the one-day log conditional moment generating function

Lemma 2

$$
E_{t-1}^Q\left[\exp\left(-u\varepsilon_t\right)\right] = \exp\left(\Psi_t\left(\nu_t + u\right) - \Psi_t\left(\nu_t\right)\right)
$$

Proof.

$$
\begin{aligned}
E_{t-1}^Q\left[\exp\left(-u\varepsilon_t\right)\right] &= E^P\left[\left(\frac{\frac{dQ}{dP}|F_t}{\frac{dQ}{dP}|F_{t-1}}\right)\exp(-u\varepsilon_t)|F_{t-1}\right] \\
&= E^P\left[\exp\left(-\nu_t\varepsilon_t - \Psi_t\left(\nu_t\right)\right)\exp(-u\varepsilon_t)|F_{t-1}\right] \\
&= \exp\left(\Psi_t\left(\nu_t + u\right) - \Psi_t\left(\nu_t\right)\right)
\end{aligned}
$$

∎

From this lemma, if we define $\Psi_t^Q\left(u\right)$ to be the log conditional moment generating function under the risk neutral probability measure, then we have

$$
\Psi_t^Q\left(u\right) = \Psi_t\left(\nu_t + u\right) - \Psi_t\left(\nu_t\right) \tag{2.7}
$$

From this we can derive

$$
E_{t-1}^Q\left[\varepsilon_t\right] = \left.\frac{\partial \Psi_t^Q\left(-u\right)}{\partial u}\right|_{u=0} = -\Psi_t'\left(\nu_t\right)
$$

Define the risk neutral innovation

$$
\varepsilon_t^* \equiv \varepsilon_t - E_{t-1}^Q\left[\varepsilon_t\right] = \varepsilon_t + \Psi_t'\left(\nu_t\right) \tag{2.8}
$$

10

The risk-neutral log-conditional moment generating function of ε_t^*, labeled $\Psi_t^{Q*}(u)$, is then

$$\Psi_t^{Q*}(u) = -u\Psi_t'(\nu_t) + \Psi_t^Q(u) \tag{2.9}$$

We are now ready to show the following

Proposition 3 *If the physical conditional distribution of ε_t is an infinitely divisible distribution with finite second moment, then the risk-neutral conditional distribution of ε_t^* is also an infinitely divisible distribution with finite second moment.*

Proof. See the Appendix. ∎

In the special case of the normal distribution we get simply

$$\varepsilon_t^* = \varepsilon_t + \Psi_t'(\nu_t) = \varepsilon_t + \mu_t - r_t$$

and $\Psi_t^{Q*}(u) = \frac{1}{2}\sigma_t^2 u^2$ so that the risk-neutral innovations are normal and correspond to the physical innovations shifted by the equity risk premium. In the more general case, the relationship between physical and risk-neutral innovations is not necessarily this simple.

Because of the one-to-one mapping between moment generating functions and distribution functions, the proposition can be used to derive specific parametric risk-neutral distributions consistent with the parametric physical distributions assumed by the researcher.

2.5 Characterizing the risk-neutral conditional variance

The conditional risk-neutral variance, σ_t^{*2}, is of particular interest in the dynamic heteroskedastic models we consider. It can be obtained by taking the second derivative of the risk neutral log moment generating function $\Psi_t^{Q*}(u)$ and evaluating it at $u = 0$. Using equations (2.9) and (2.7) we get

$$\sigma_t^{*2} = \left.\frac{\partial^2 \Psi_t^{Q*}(-u)}{\partial u^2}\right|_{u=0} = \Psi_t''(\nu_t)$$

Recall that by definition, the conditional variance under the physical measure is $\sigma_t^2 = \Psi_t''(0)$. Thus in general we have the following relationship between the (one day ahead) conditional variances under the two measures

$$\sigma_t^{*2} = \sigma_t^2 \frac{\Psi_t''(\nu_t)}{\Psi_t''(0)}$$

For conditionally normal returns, we have $\Psi_t(u) = \frac{1}{2}\sigma_t^2 u^2$ and $\nu_t = (\mu_t - r_t)/\sigma_t^2$, so that $\Psi_t''(\nu_t) = \Psi_t''(0)$ and thus $\sigma_t^{*2} = \sigma_t^2$, but this will not generally be the case for non-normal distributions. Non-normality drives an additional wedge between the physical and risk-neutral

conditional variances. Interestingly, this phenomenon is often observed empirically, as physical volatility measures from historical returns are systematically lower than risk-neutral volatilities implied from options. See for example Carr and Wu (2007).

We can use our results to provide some more insight into this wedge between one day ahead physical and risk-neutral conditional variances. Consider the following approximation to the risk-neutral variance

$$\sigma_t^{*2} = \Psi_t''(\nu_t) \approx \Psi_t''(0) + \Psi_t'''(0)\nu_t + \frac{\Psi_t''''(0)}{2}\nu_t^2$$

Denoting conditional skewness by $skew_t$ and conditional excess kurtosis by $kurt_t$, we have $\Psi_t'''(0) = -skew_t\sigma_t^3$ and $\Psi_t''''(0) = kurt_t\sigma_t^4$. Therefore

$$\sigma_t^{*2} \approx \sigma_t^2 - skew_t\sigma_t^3\nu_t + \frac{kurt_t}{2}\sigma_t^4\nu_t^2 \tag{2.10}$$

From (2.5), ν_t can be thought of as a modified Sharpe ratio, and will generally be positive. Therefore, from (2.10), the risk neutral variance will always be larger than the historical variance if conditional skewness is negative and/or excess kurtosis is positive.

Furthermore, we can characterize the risk-neutral conditional variance dynamic. As an example, start from the simple GARCH(1,1) dynamic of Bollerslev (1986) for the physical conditional variance

$$\sigma_t^2 = \beta_0 + \beta_1\sigma_{t-1}^2 + \beta_2\sigma_{t-1}^2\varepsilon_{t-1}^2 \tag{2.11}$$

which can be shown to lead to the risk-neutral variance dynamic

$$\sigma_t^{*2} = \beta_{0,t} + \beta_{1,t}\sigma_{t-1}^{*2} + \beta_{2,t}\left(\varepsilon_{t-1}^* - \Psi'(\nu_{t-1})\right)^2$$

where

$$\beta_{0,t} = \beta_0\frac{\Psi_t''(\nu_t)}{\Psi_t''(0)}, \beta_{1,t} = \beta_1\frac{\Psi_t''(\nu_t)}{\Psi_t''(0)}\frac{\Psi_{t-1}''(0)}{\Psi_{t-1}''(\nu_{t-1})}, \beta_{2,t} = \beta_2\frac{\Psi_t''(\nu_t)}{\Psi_t''(0)}$$

Under normality $\beta_{0,t} = \beta_0$, $\beta_{1,t} = \beta_1$, and $\beta_{2,t} = \beta_2$, and therefore

$$\sigma_t^{*2} = \beta_0 + \beta_1\sigma_{t-1}^{*2} + \beta_2\left(\varepsilon_{t-1}^* - \Psi'(\nu_{t-1})\right)^2 \tag{2.12}$$

Taking into account that under normality we also have $\sigma_t^{*2} = \sigma_t^2$, this can be re-written as

$$\sigma_t^2 = \beta_0 + \beta_1\sigma_{t-1}^2 + \beta_2\left(\varepsilon_{t-1}^* - \Psi'(\nu_{t-1})\right)^2 \tag{2.13}$$

Note that (2.13) can also be derived by using the expression for the risk-neutral innovation (2.8) in (2.11). This derivation does not depend on normality. Therefore, (2.13) holds in general but

it is only under normality that the risk-neutral variance (2.12) follows the same dynamic with the same coefficients, which is consistent with the finding that $\sigma_t^{*2} = \sigma_t^2$ for conditionally normal returns. We will discuss the implications of conditionally non-normal returns further below, and give explicit examples of non-normal distributions that generate the interesting and important empirical feature that physical and risk-neutral one day ahead conditional variances differ.

2.6   Characterizing Risk-Neutral Conditional Skewness

We can also derive a useful result on risk-neutral skewness. Using

$$\Psi_t'''(0) = -skew_t\sigma_t^3 \text{ and } \Psi_t'''(\nu_t) = -skew_t^*\sigma_t^{*3}$$

as well as

$$\Psi_t'''(\nu_t) \approx \Psi_t'''(0) + \Psi_t''''(0)\,\nu_t \text{ and } \Psi_t''''(0) = kurt_t\sigma_t^4$$

we get that

$$-skew_t^*\sigma_t^{*3} \approx -skew_t\sigma_t^3 + kurt_t\sigma_t^4\nu_t$$
$$skew_t^* \approx skew_t\left(\frac{\sigma_t}{\sigma_t^*}\right)^3 - kurt_t\frac{\sigma_t^4\nu_t}{\sigma_t^{*3}}$$

Note that for the empirically relevant case where $\sigma_t \leq \sigma_t^*$, we have $skew_t\left(\frac{\sigma_t}{\sigma_t^*}\right)^3 \leq skew_t$. Therefore $skew_t^* \leq skew_t$ for the empirically relevant case where the price of risk $\nu_t \geq 0$ and $kurt_t \geq 0$.

3   Generalized EMMs and Option Price Bounds

While the one-shock stock price processes in Section 2.1, and the GARCH processes nested in it, imply an incomplete-markets setup, we do obtain a unique price conditional on the choice of Radon-Nikodym derivative. Clearly therefore there have to be other valid prices corresponding to other choices of Radon-Nikodym derivative. We now characterize EMMs corresponding to other classes of Radon-Nikodym derivatives.

3.1   Generalized EMMs in GARCH models

We use the dynamic of the stock price process under the physical measure in (2.1), with $\Psi_t(u)$ the natural logarithm of the moment generating function. In order to allow for as much generality as possible while still staying in our framework we define a class of Radon-Nikodym derivatives

defined by a general log-MGF under Q, call it $\Omega_t(u)$. We then show which restrictions need to be placed on $\Omega_t(u)$ in order for it to result in a proper EMM.

First, define the following candidate Radon-Nikodym derivative for a given predetermined sequence of log moment generating functions $\{\Omega_t(u)\}$, which is F_{t-1} adapted,

$$\frac{dQ}{dP}\bigg|F_t = \prod_{j=1}^{t} \frac{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_j + \Omega_j(-iu)\right) du}{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_j + \Psi_j(-iu)\right) du} \tag{3.1}$$

Lemma 3 $\frac{dQ}{dP}\big|F_t$ *is a Radon-Nikodym derivative*

Proof. We need to show that $\frac{dQ}{dP}\big|F_t > 0$. For each j, $\exp\left(\Omega_j(-iu)\right)$ is a characteristic function which is absolutely integrable over $(-\infty, +\infty)$. Using the inversion formula (Lukacs (1970, p. 33)), $q_j(\varepsilon_j) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_j + \Omega_j(-iu)\right) du$ is the corresponding density function. Similarly $p_j(\varepsilon_j) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_j + \Psi_j(-iu)\right) du$ is a density function. Therefore

$$\frac{dQ}{dP}\bigg|F_t = \prod_{j=1}^{t} \frac{q_j(\varepsilon_j)}{p_j(\varepsilon_j)}$$

We have $\frac{dQ}{dP}\big|F_t > 0$ because density functions are always positive. We also need to show $E_0^P\left[\frac{dQ}{dP}\big|F_t\right] = 1$. We have

$$E_0^P\left[\frac{dQ}{dP}\bigg|F_t\right] = E_0^P\left[\prod_{j=1}^{t} \frac{q_j(\varepsilon_j)}{p_j(\varepsilon_j)}\right].$$

Using the law of iterated expectations we have

$$E_0^P\left[\frac{dQ}{dP}\bigg|F_t\right] = E_0^P\left[E_1^P...E_{t-1}^P \prod_{j=1}^{t} \frac{q_j(\varepsilon_j)}{p_j(\varepsilon_j)}\right]$$

$$= E_0^P\left[E_1^P...E_{t-2}^P \prod_{j=1}^{t-1} \frac{q_j(\varepsilon_j)}{p_j(\varepsilon_j)} E_{t-1}^P \frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)}\right]$$

Note

$$E_{t-1}^P \frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)} = \int \frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)} p_t(\varepsilon_t) d\varepsilon_t$$

Therefore $E_{t-1}^P \frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)} = \int q_t(\varepsilon_t) d\varepsilon_t = 1$ and

$$E_0^P\left[\frac{dQ}{dP}\bigg|F_t\right] = E_0^P\left[E_1^P...E_{t-2}^P \prod_{j=1}^{t-1} \frac{q_j(\varepsilon_j)}{p_j(\varepsilon_j)}\right]$$

Iteratively using this result we get

$$E_0^P \left[\left. \frac{dQ}{dP} \right| F_t \right] = E_0^P \left[\frac{q_1(\varepsilon_1)}{p_1(\varepsilon_1)} \right] = 1$$

and the lemma obtains. ∎

We are now ready to show the restriction required on $\Omega_t(u)$ so that we can specify an EMM using this Radon-Nikodym derivative.

Proposition 4 *The probability measure Q defined by the Radon-Nikodym derivative in (3.1) is an EMM if and only if*

$$\Omega_t(-1) - \gamma_t + \phi_t \sigma_t^2 = 0 \tag{3.2}$$

where $\phi_t = \dfrac{\mu_t - r_t}{\sigma_t^2}$.

Proof. We need $E^Q \left[\left. \frac{S_t}{B_t} \right| F_{t-1} \right] = \frac{S_{t-1}}{B_{t-1}}$ or equivalently $E^Q \left[\left. \frac{S_t}{S_{t-1}} \middle/ \frac{B_t}{B_{t-1}} \right| F_{t-1} \right] = 1$. We have

$$E^Q \left[\left. \frac{S_t}{S_{t-1}} \middle/ \frac{B_t}{B_{t-1}} \right| F_{t-1} \right] = E^P \left[\left(\frac{\frac{dQ}{dP} | F_t}{\frac{dQ}{dP} | F_{t-1}} \right) \frac{S_t}{S_{t-1}} \middle/ \frac{B_t}{B_{t-1}} \right| F_{t-1} \right]$$

$$\begin{aligned}
&= E^P \left[\left(\frac{\frac{dQ}{dP} | F_t}{\frac{dQ}{dP} | F_{t-1}} \right) \frac{S_t}{S_{t-1}} \exp(-r_t) \middle| F_{t-1} \right] \\
&= E^P \left[\frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)} \exp(\mu_t - \gamma_t + \varepsilon_t) \exp(-r_t) \middle| F_{t-1} \right] \\
&= \exp(\mu_t - r_t - \gamma_t) E^P \left[\exp(\varepsilon_t) \frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)} | F_{t-1} \right] \\
&= \exp(\mu_t - r_t - \gamma_t) \int \exp(\varepsilon_t) \frac{q_t(\varepsilon_t)}{p_t(\varepsilon_t)} p_t(\varepsilon_t) d\varepsilon_t \\
&= \exp(\mu_t - r_t - \gamma_t) \int \exp(\varepsilon_t) q_t(\varepsilon_t) d\varepsilon_t \\
&= \exp(\mu_t - r_t - \gamma_t + \Omega_t(-1))
\end{aligned}$$

since by definition $\Omega_t(u)$ is the log-MGF which corresponds to the density $q_t(\varepsilon_t)$. By taking logs the lemma obtains. ∎

This result shows that a Radon-Nikodym derivative can be defined such that any log-MGF $\Omega_t(u)$ satisfying equation (3.2) will provide a suitable EMM. The result implies that a wide class of EMMs are possible.

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3.2 Nesting the Linear EMM

We now demonstrate how the class of Radon-Nikodym derivatives in Section 2.2, which is linear in the stock return innovation, is nested in the class of Radon-Nikodym derivatives discussed above. For a given sequence, $\{\nu_t\}$, we restrict the function $\Omega_t(u)$ in (3.1) as follows

$$\Omega_t(u) = \Psi_t(u + \nu_t) - \Psi_t(\nu_t) \tag{3.3}$$

This is known as the Esscher transform. Note that this corresponds to $\Psi_t^Q(u)$ defined in (2.7). While other transforms could be chosen, such as for example $\Omega_t(u) = \nu_t \Psi_t(u)$, this particular choice is convenient from an analytical perspective.[11] The condition (3.2) becomes

$$\Psi_t(\nu_t - 1) - \Psi_t(\nu_t) - \gamma_t + \phi_t \sigma_t^2 = 0 \tag{3.4}$$

which is equal to (2.3). Substituting (3.3) in (3.1) gives

$$
\begin{aligned}
\frac{\frac{dQ}{dP}|F_t}{\frac{dQ}{dP}|F_{t-1}} &= \frac{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_t + \Psi_t(-iu + \nu_t) - \Psi_t(\nu_t)\right) du}{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_t + \Psi_t(-iu)\right) du} \\
&= \exp\left(-\Psi_t(\nu_t)\right) \frac{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_t + \Psi_t(-iu + \nu_t)\right) du}{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_t + \Psi_t(-iu)\right) du} \\
&= \exp\left(-\nu_t\varepsilon_t - \Psi_t(\nu_t)\right) \frac{\int_{-\infty}^{+\infty} \exp\left(-i(u + i\nu_t)\varepsilon_t + \Psi_t(-i(u + i\nu_t))\right) du}{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_t + \Psi_t(-iu)\right) du} \\
&= \exp\left(-\nu_t\varepsilon_t - \Psi_t(\nu_t)\right) \frac{\int_{-\infty}^{+\infty} \exp\left(-iu^*\varepsilon_t + \Psi_t(-iu^*)\right) du^*}{\int_{-\infty}^{+\infty} \exp\left(-iu\varepsilon_t + \Psi_t(-iu)\right) du} \\
&= \exp\left(-\nu_t\varepsilon_t - \Psi_t(\nu_t)\right)
\end{aligned}
$$

where we have used the fact that $i^2 = -1$, as well as a change of measure, $u^* = u + i\nu_t$. Note that his result corresponds exactly to the assumption on the Radon-Nikodym derivative in (2.2).

We have thus demonstrated how the class of Radon-Nikodym derivatives in (2.2) obtains as a special case of the general characterization of the class of Radon-Nikodym derivatives in (3.1). In Section 2.4 above, and below in Section 5, we demonstrate that this special case is of great interest because it allows us to characterize the risk-neutral dynamics in closed form for a large class of return innovations. Such characterizations are as a rule not possible with the more general class of Radon-Nikodym derivatives. However, given that Radon-Nikodym derivatives

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[11] For applications of the Esscher transform in option valuation, see Buhlmann, Delbaen, Embrechts and Shiryaev (1996, 1998), Gerber and Shiu (1994), and Siu, Tong and Yang (2004). See Dai and Singleton (2006) for an application to term structure models.

typically used in empirical work are of the form in (2.2), and that the resulting risk-neutralizations have some empirical shortcomings, it may be of interest to analyze richer specifications of the Radon-Nikodym derivative.

3.3 A Quadratic EMM Under Conditional Normality

We now analyze a somewhat more general case that still allows for some analytical results. Specifically, we analyze the case of a quadratic rather than linear EMM, but we restrict ourselves to normally distributed innovations.

For a given sequence $\{\nu_{1,t}, \nu_{2,t}\}$, consider the following candidate Radon-Nikodym derivative

$$\frac{dQ}{dP}\bigg|\, F_t = \exp\left(-\sum_{i=1}^{t}\left(\nu_{1,i}\varepsilon_i + \nu_{2,i}\varepsilon_i^2 + g\left(\nu_{1,i}, \nu_{2,i}, \sigma_i^2\right)\right)\right) \tag{3.5}$$

By solving the EMM equation, $E^Q\left[\dfrac{S_t}{S_{t-1}}\bigg/\dfrac{B_t}{B_{t-1}}\bigg|\, F_{t-1}\right] = 1$, we can show that the probability measure Q defined by the Radon-Nikodym derivative in (3.5) is an EMM if and only if

$$g\left(\nu_{1,i}, \nu_{2,i}, \sigma_i^2\right) = \frac{1}{2}\left(\nu_{1,i}^2 \sigma_i^{*2} - \ln\left(\sigma_i^2/\sigma_i^{*2}\right)\right), \text{ where} \tag{3.6}$$

$$\sigma_i^{*2} = Var_{i-1}^Q\left(\varepsilon_i\right) = \frac{\sigma_i^2}{1 + 2\nu_{2,i}\sigma_i^2}, \text{ and} \tag{3.7}$$

$$\nu_{1,i} = \left[\frac{\mu_i}{\sigma_i^2} - \frac{r_i}{\sigma_i^{*2}}\right] + 2\left(\mu_i - \frac{1}{2}\sigma_i^2\right)\nu_{2,i} \tag{3.8}$$

An interesting feature of this EMM is that we get a wedge between the physical and risk-neutral variance–an empirically observed fact–even when assuming conditional normality of returns. In this case the wedge is driven by the quadratic term, $\nu_{2,t}$, in the pricing kernel. Recall that in section 2.5 above a wedge was created by non-normality in the conditional return distribution.

Note that we have two EMM parameters, $\nu_{1,i}$ and $\nu_{2,i}$, but only one equation defining $\nu_{2,i}$ as a function of $\nu_{1,i}$. In order to complete the model we could impose that the proportional wedge between σ_t^{*2} and σ_t^2 is constant. If we for example set $\sigma_t^2/\sigma_t^{*2} = \pi_\sigma$, we get $\nu_{2,t} = \frac{1}{2}\left(\pi_\sigma - 1\right)/\sigma_t^2$.

Next we consider how this quadratic case fits into our general setup discussed in Section 3.1.

Since we are working with normal innovations, we can use the inversion formula to write

$$\frac{\frac{dQ}{dP}|F_t}{\frac{dQ}{dP}|F_{t-1}} = \frac{q_t\left(\varepsilon_t\right)}{p_t\left(\varepsilon_t\right)} = \frac{\frac{1}{\sigma_t^* \sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(\varepsilon_t + \delta_t^*)^2}{\sigma_t^{*2}}\right)}{\frac{1}{\sigma_t \sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{\varepsilon_t^2}{\sigma_t^2}\right)} = \exp\left(-\frac{1}{2}\frac{(\varepsilon_t + \delta_t^*)^2}{\sigma_t^{*2}} + \frac{1}{2}\frac{\varepsilon_t^2}{\sigma_t^2} + \ln\left(\frac{\sigma_t}{\sigma_t^*}\right)\right)$$

$$= \exp\left(-\nu_{1,t}\varepsilon_t - \nu_{2,t}\varepsilon_t^2 + \ln\left(\frac{\sigma_t}{\sigma_t^*}\right) - \frac{\delta_t^{*2}}{2\sigma_t^{*2}}\right)$$

where δ_t^* is the risk-neutral mean of ε_t and where

$$\nu_{1,t} = \frac{\delta_t^*}{\sigma_t^{*2}}, \text{ and } \nu_{2,t} = \frac{1}{2}\left(\frac{1}{\sigma_t^{*2}} - \frac{1}{\sigma_t^2}\right)$$

From normality we have that $\Omega_t\left(-1\right) = \frac{1}{2}\sigma_t^{*2} - \delta_t^{*2}$ and from the EMM condition in (3.2) we have that $\Omega_t\left(-1\right) = \mu_t - r_t - \frac{1}{2}\sigma_t^2$. These equations provide an expression for the risk-neutral mean of ε_t in the quadratic model

$$\delta_t^* = \mu_t - r_t + \frac{1}{2}\left(\sigma_t^{*2} - \sigma_t^2\right) \tag{3.9}$$

Using this equation for δ_t^* and the equation for $\nu_{2,t}$ in the equation for $\nu_{1,t}$ will yield (3.8).

We have thus shown how in the normal case the quadratic EMM in (3.5) is a special case of the general class of EMMs defined by (3.1). Note also that by setting $\nu_{2,t} = 0$, we obtain the affine EMM as a special case.

3.4 Market Incompleteness and Bounds on Option Prices

Market incompleteness results in a wide range of available Radon-Nikodym derivatives and thus multiple EMMs and option prices. In order to illustrate this incompleteness consider Figure 1. We use the linear and quadratic EMMs to compute the price of a one-month-to-maturity, at-the-money call option with an underlying asset price of 100. We assume a risk-free rate of 5%, an underlying mean asset return of 10% and a physical asset volatility of 20% per year. In the quadratic EMM we let the ratio of the physical to risk-neutral variance, $\sigma_t^2/\sigma_t^{*2} = \pi_\sigma$ vary from 0.5 to 1. Figure 1 shows how the option price from the quadratic EMM depends critically on π_σ and thus $\nu_{2,t}$ in (3.5). The horizontal line shows the option price from the linear EMM where $\pi_\sigma = 1$ and $\nu_{2,t} = 0$. Figure 1 shows that the range of option prices can be wide even when staying within the quadratic class of EMMs. This illustrates the potential of non-linear EMMs to explain outstanding empirical puzzles such as the high prices of deep out-of-the-money index put options.

The literature on option pricing bounds provides a way to quantify market incompleteness.

Key early papers in this literature include Perrakis and Ryan (1984), Levy (1985), and Ritchken (1985) who all applied single-period models. Perrakis (1986) and Ritchken and Kuo (1988) extended this work to a multi-period setting, and Constantinides, Jackwerth and Perrakis (2008) contain a recent application to S&P500 index options. These papers proceed by considering a portfolio of an option, an underlying asset and a risk-free bond and derive bounds on the option price without assuming a particular EMM but instead relying only on the principle of stochastic dominance. The bounds are defined so that observing an option price outside the bounds would induce a stochastically dominating trading strategy.

While the work in this literature has evolved to allow for trading costs and other frictions (see Constantinides and Perrakis, 2002, 2007) until recently the results were developed in an i.i.d. setting, thus ruling out the GARCH effects considered in this paper. However, current work by Oancea and Perrakis (2007) extends the stochastic dominance approach to derive intervals of admissible option prices using bounds allowing for GARCH effects. But in contrast with the i.i.d. case it is necessary in the GARCH case to assume that the representative investor has constant relative risk aversion.

The recent so-called good-deal bounds approach of Cochrane and Saa-Requejo (2000) presents another interesting venue for generating option pricing bounds.[12] Good-deal bounds are derived using a distance measure between a given stochastic discount factor (SDF) and a benchmark SDF. This approach has been adapted to option pricing under continuous-time stochastic volatility by Bondarenko and Longarela (2004). We can show that it is possible in the discrete GARCH framework to derive good-deal bounds on option prices when using a quadratic EMM.[13]

4   The valuation of European style contingent claims

In a general return model with time-varying conditional mean and volatility and non-normal shocks, we have characterized conditions under which there exists an EMM Q that makes the stock discounted by the riskless asset a martingale.

We now turn our attention to the pricing of European style contingent claims. Existing papers on the pricing of contingent claims in a discrete-time infinite state space setup, such as the literature on GARCH option pricing in Duan (1995), Amin and Ng (1993) and Heston and Nandi (2000) value such contingent claims by making an assumption on the bivariate distribution of the stock return and the endowment, or an equivalent assumption. While this approach, which most often amounts to the characterization of the equilibrium that supports the pricing, is an elegant way to deal with the incompleteness that characterizes these markets, we argue that it

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[12]See Bernardo and Ledoit (2000) for a related approach.

[13]This result is available from the authors upon request.

is not strictly necessary to characterize the equilibrium. Instead, we adopt an approach which is more prevalent in the continuous-time literature, and proceed to pricing derivatives using a no-arbitrage argument alone.

To understand our approach, the analogy with option valuation for the stochastic volatility model of Heston (1993a) is particularly helpful. In this incomplete markets setting, an infinity of no-arbitrage contingent claims prices exist, one for every different specification of the price of risk. When one fixes the price of volatility risk, however, there is a unique no-arbitrage price. For the purpose of option valuation, one can simply pick a price of volatility risk, and the resulting valuation exercise is purely mechanical.

The question whether a particular price of risk is reasonable is of substantial interest in its own right, and an analysis of the representative agent utility function that support a particular price of risk is very valuable. However, this question can be analyzed separately from the option valuation problem. For the heteroskedastic discrete-time models we consider, a similar remark applies. The link between our approach and the utility-based approach in Brennan (1979), Rubinstein (1976) and Duan (1995) is that assumptions on the utility function are implicit in the specification of the risk premium in the return dynamic in our case.[14] The representative agent preferences underlying this assumption are of interest, but it is not necessary to analyze them in order to value options.

We have already found an EMM Q. We therefore want to demonstrate that the price at time t is defined as

$$C_t = E^Q \left[\frac{C_T(S_T)}{B_T} B_t \,\middle|\, F_t \right].$$

The proof proceeds in a number of steps and requires defining a number of concepts that are well-known in the literature. Fortunately, even though our methodology closely follows the continuous-time case, we economize on the number of technical conditions in the continuous-time setup, such as admissibility, and avoid the concepts of local martingale and super martingale. The reason is that the integration over an infinite number of trading times in the continuous-time case is replaced by a finite sum over the trading days in discrete time.

Definitions

1. We denote by η_t, δ_t and ψ_t the units of the stock, the contingent claim and the bond held at date t. We refer to the F_t predictable processes η_t, δ_t and ψ_t as investment strategies.

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[14]See Bick (1990) and He and Leland (1993) for a discussion of assumptions on the utility function implicit in the specification of the return dynamic for the market portfolio. We proceed along the lines of Jacod and Shiryaev (1998), and Shiryaev (1999).

2. The value process

$$V_t = \eta_t S_t + \delta_t C_t + \psi_t B_t$$

describes the total dollar amount available for investments at date t.

3. The gain process

$$G_t = \sum_{i=0}^{t-1} \eta_i(S_{i+1} - S_i) + \sum_{i=0}^{t-1} \delta_i(C_{i+1} - C_i) + \sum_{i=0}^{t-1} \psi_i(B_{i+1} - B_i).$$

captures the total financial gains between dates 0 and t.

4. We call the process $\{\eta_t, \delta_t, \psi_t\}_{t=0}^{T-1}$ a self financing strategy if and only if $V_t = G_t \; \forall t = 1, ..., T$.

5. The definition of an arbitrage opportunity is standard: we have an arbitrage opportunity if a self financing strategy exists with either $V_0 < 0$, $V_T \geq 0$ a.s. or $V_0 \leq 0$, $V_T > 0$ a.s.

6. We denote the discounted stock price at time t as $S_t^B = \frac{S_t}{B_t}$ and the discounted contingent claim as $C_t^B = \frac{C_t}{B_t}$. Similarly, the discounted value process is denoted $V_t^B = \frac{V_t}{B_t}$ and the discounted gain process $G_t^B = \frac{G_t}{B_t}$.

Note that for a self financing strategy, we have $V_t^B = G_t^B$ because $V_t = G_t$ and $B_t > 0$. Furthermore, we can show the following.

Lemma 4 *For a self financing strategy we have*

$$G_t^B = \sum_{i=0}^{t-1} \eta_i(S_{i+1}^B - S_i^B) + \sum_{i=0}^{t-1} \delta_i(C_{i+1}^B - C_i^B) \qquad \forall t = 1, ..., T$$

Proof. The proof involves straightforward but somewhat cumbersome algebraic manipulations of the above definitions. See the Appendix for the details. ∎

We know that under the EMM we defined, the stock discounted by the risk free asset is a martingale. We now need to show that the contingent claims prices obtained by computing the expected value of the final payoff discounted by the risk free asset also constitute a martingale under this EMM.

Lemma 5 *The stochastic process defined by the discounted values of the candidate contingent claims prices is an F_t martingale under the EMM.*

21

Proof. We defined our candidate process for the contingent claims price under the EMM as $C_t = E^Q \left[\frac{C_T(S_T)}{B_T} B_t \middle| F_t \right]$. The process for the discounted values of the contingent claims prices is then defined as

$$C_t^B \equiv \frac{C_t}{B_t} = E^Q \left[\frac{C_T(S_T)}{B_T} \middle| F_t \right]$$

We use the fact that the conditional expectation itself is a Q martingale. This in turn follows from the law of iterated expectations and the European style payoff function. Taking conditional expectations with respect to F_s on both sides of the above equation yields

$$E^Q \left[\frac{C_t}{B_t} \middle| F_s \right] = E^Q \left[E^Q \left[\frac{C_T(S_T)}{B_T} \middle| F_t \right] \middle| F_s \right] \qquad \forall t > s$$

Now using the law of iterated expectations we get

$$E^Q \left[\frac{C_t}{B_t} \middle| F_s \right] = E^Q \left[\frac{C_T(S_T)}{B_T} \middle| F_s \right] = \frac{C_s}{B_s} = C_s^B \qquad \forall t > s$$

which gives the desired result. ∎

Lemma 6 *Under the EMM defined by (2.2), the discounted gain process is a martingale.*

Proof. Under the EMM Q, the process $\left\{ S_t^B \right\}_{t=1}^T$ is a Q martingale. Using a standard property of martingales the process defined as $SS_t^B = \sum_{i=0}^{t-1} \eta_i (S_{i+1}^B - S_i^B)$ then is a Q martingale, since the investment strategy η_t is included in the information set.[15] Furthermore, from Lemma 5 we get that $\left\{ C_t^B \right\}_{t=1}^T$ is also a Q martingale. Then using the fact that δ_t is an F_t predetermined process and using the same martingale property as above we get that the process $CC_t^B = \sum_{i=0}^{t-1} \delta_i (C_{i+1}^B - C_i^B)$ is a Q martingale. Then since from Lemma 4 the discounted gain process $\left\{ G_t^B \right\}_{t=1}^T$ is the sum of two Q martingales, SS_t^B and CC_t^B, it is itself a Q martingale. ∎

At this stage, we have all the ingredients to show the following result.

Proposition 5 *If we have an EMM that makes the discounted price of the stock a martingale, then defining the price of any contingent claim as the expected value of its payoff, taken under this EMM and discounted at the riskless interest rate, constitutes a no-arbitrage price.*

Proof. From Lemma 6 G_t^B is a Q martingale. Because we are considering self financing strategies we get that V_t^B is a martingale. We prove the absence of arbitrage by contradiction. If we assume the existence of an arbitrage opportunity, then there exists a self financing strategy with type 1

---

[15]Note that because we are working in discrete time there is no need to investigate the integrability of SS_t^B.

arbitrage ($V_0 < 0, V_T \geq 0$ a.s.) or type 2 arbitrage ($V_0 \leq 0, V_T > 0$ a.s.). Both cases lead to a clear contradiction. Consider type 1 arbitrage: we start from the existence of a self financing strategy with $V_0 < 0$ that ends up with a positive final value. $V_0 < 0$ implies that $V_0^B < 0$ since the numeraire is always positive by definition. Also since $V_T \geq 0$ we have $V_T^B \geq 0$. Taking expectations and using the fact that V_t^B is a Q martingale yields $V_0^B = E_0^Q[V_T^B] \geq 0$. This is a contradiction because we assumed that we start with a negative value $V_0 < 0$. A similar argument works for type 2 arbitrage. Thus, the C_t from the EMM Q must be a no-arbitrage price. ∎

In summary, we have demonstrated that in a discrete-time infinite state space setting, if we have an EMM that makes the underlying asset price a martingale, then the expected value of the payoff of the contingent claim taken under this EMM, discounted at the riskless asset, is a no-arbitrage price. In Section 2.2, we derived such an EMM. Altogether, we have therefore demonstrated that for any contingent claim paying a final payoff $C_T(S_T)$ the current price C_t can be computed as

$$C_t = E^Q \left[\frac{C_T(S_T)}{B_T} B_t \,\middle|\, F_t \right].$$

5   Important special cases

In this section we demonstrate how a number of important existing models are nested in our setup, using the class of linear Radon-Nikodym derivatives in (2.2). We first consider various specifications of the equity risk premium in the conditional normal setting. We then consider two conditional non-normal specifications relying on inverse Gaussian shocks and Poisson jumps respectively.

5.1   Flexible risk premium specifications

One of the advantages of our approach is that we can allow for general specifications of the time-varying equity risk premium. Here we discuss some potentially interesting ways to specify the risk premium in the return process for the underlying asset. In order to demonstrate the link with the available literature and for computational simplicity, we assume conditional normal returns, although this assumption is by no means necessary.

The conditional normal models in the Duan (1995) and Heston and Nandi (2000) models are special cases of our set-up. In our notation, Duan (1995) assumes

$$r_t = r, \text{ and } \mu_t = r + \lambda \sigma_t$$

which in our framework corresponds to a Radon-Nikodym derivative of

$$\frac{dQ}{dP}\bigg| F_t = \exp\left(-\sum_{i=1}^{t}\left(\frac{\varepsilon_i}{\sigma_i}\lambda + \frac{1}{2}\lambda^2\right)\right)$$

and risk neutral innovations of the form

$$\varepsilon_t^* = \varepsilon_t + \lambda\sigma_t.$$

Heston and Nandi (2000) instead assume

$$r_t = r, \text{ and } \mu_t = r + \lambda\sigma_t^2 + \frac{1}{2}\sigma_t^2$$

which in our framework corresponds to a Radon-Nikodym derivative of

$$\frac{dQ}{dP}\bigg| F_t = \exp\left(-\sum_{i=1}^{t}\left(\left(\lambda + \frac{1}{2}\right)\varepsilon_i + \frac{1}{2}\left(\lambda + \frac{1}{2}\right)^2\sigma_i^2\right)\right)$$

and risk neutral innovations of the form

$$\varepsilon_t^* = \varepsilon_t + \lambda\sigma_t^2 + \frac{1}{2}\sigma_t^2.$$

However, many empirically relevant cases are not covered by existing theoretical results. For example, in the original ARCH-M paper, Engle, Lilien and Robins (1987) find the strongest empirical support for a risk premium specification of the form

$$\mu_t = r_t + \lambda\ln\left(\sigma_t\right) + \tfrac{1}{2}\sigma_t^2$$

which cannot be used for option valuation using the available theory. In our framework it simply corresponds to a Radon-Nikodym derivative of

$$\frac{dQ}{dP}\bigg| F_t = \exp\left(-\sum_{i=1}^{t}\left(\frac{\lambda\ln\left(\sigma_i\right) + \frac{1}{2}\sigma_i^2}{\sigma_i^2}\varepsilon_i + \frac{1}{2}\left(\frac{\lambda\ln\left(\sigma_i\right) + \frac{1}{2}\sigma_i^2}{\sigma_i^2}\right)^2\sigma_i^2\right)\right)$$

and risk neutral innovations

$$\varepsilon_t^* = \varepsilon_t + \lambda\ln\left(\sigma_t\right) + \tfrac{1}{2}\sigma_t^2$$

Our approach allows for option valuation under such specifications whereas the existing literature does not.

5.2 Conditionally inverse Gaussian returns

Christoffersen, Heston and Jacobs (2006) analyze a GARCH model with an inverse Gaussian innovation, $y_t \sim IG(\sigma_t^2/\eta^2)$. We can write their return dynamic as

$$R_t = r + \left(\zeta + \eta^{-1}\right)\sigma_t^2 + \varepsilon_t, \text{ where} \tag{5.1}$$

$$\varepsilon_t = \eta y_t - \eta^{-1}\sigma_t^2 \tag{5.2}$$

and where the conditional return variance, σ_t^2, is of the GARCH form. The inverse Gaussian belongs to the class of infinitely divisible distributions, which yields the strict convexity in Proposition 2, and the other conditions of Proposition 2 are also satisfied.

From the MGF of an inverse Gaussian variable, we can derive the conditional log MGF

$$\Psi_t(u) = \left(u + \frac{1 - \sqrt{1 + 2u\eta}}{\eta}\right)\frac{\sigma_t^2}{\eta}$$

The EMM condition

$$\Psi_t(\nu_t - 1) - \Psi_t(\nu_t) - \Psi_t(-1) + \phi_t\sigma_t^2 = 0$$

is now solved by the constant

$$\nu_t = \nu = \frac{1}{2\eta}\left[\frac{\left(2 + \zeta\eta^3\right)^2}{4\zeta^2\eta^2} - 1\right], \forall t$$

which in turn implies that the EMM is given by

$$\begin{aligned}\left.\frac{dQ}{dP}\right|F_t &= \exp\left(-\sum_{i=1}^{t}\left(\nu\varepsilon_i + \left(\nu + \frac{1 - \sqrt{1 + 2\nu\eta}}{\eta}\right)\frac{\sigma_i^2}{\eta}\right)\right) \\ &= \exp\left(-\nu t\bar{\varepsilon}_t - \delta t\overline{\sigma_t^2}\right)\end{aligned}$$

where $\bar{\varepsilon}_t = \frac{1}{t}\sum_{i=1}^{t}\varepsilon_i$, $\overline{\sigma_t^2} = \frac{1}{t}\sum_{i=1}^{t}\sigma_i^2$, and $\delta = \frac{\nu}{\eta} + \frac{1 - \sqrt{1 + 2\nu\eta}}{\eta^2}$.

These expressions can be used to obtain the risk-neutral distribution from Christoffersen, Heston and Jacobs (2006) using the results in Section 2. Recall that in general the risk neutral log MGF is

$$\Psi_t^{Q*}(u) = -u\Psi_t'(\nu) + \Psi_t(\nu + u) - \Psi_t(\nu)$$

In the GARCH-IG case we can write

$$\Psi_t^{Q*}(u) = \left(u + \frac{1 - \sqrt{1 + 2u\eta^*}}{\eta^*}\right)\frac{\sigma_t^{*2}}{\eta^*}$$

where

$$\eta^* = \frac{\eta}{1 + 2\nu\eta} \text{ and } \sigma_t^{*2} = \frac{\sigma_t^2}{(1 + 2\nu\eta)^{3/2}}$$

which indicates that generally the risk-neutral variance will be different from the physical variance. The risk neutral return model can be written as

$$R_t \equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r - \Psi_t^{Q*}(-1) + \varepsilon_t^* = r + \left(\zeta^* + \eta^{*-1}\right)\sigma_t^{*2} + \varepsilon_t^*$$

where

$$\zeta^* = \frac{1 - 2\eta^* - \sqrt{1 - 2\eta^*}}{\eta^{*2}} \text{ and } \varepsilon_t^* = \eta^* y_t^* - \eta^{*-1}\sigma_t^{*2}$$

The risk neutral process thus takes the same form as the physical process, confirming Proposition 3 in Section 2.4.

5.3 Conditionally Poisson-normal jumps

Another interesting model that can be nested in our framework is the heteroskedastic model with Poisson-normal innovations in Duan, Ritchken and Sun (2005).[16] For expositional simplicity, we consider the simplest version of the model. More complex models, for instance with time-varying Poisson intensities, can also be accommodated. The conditions of Proposition 2 can again readily be verified, in part because the Poisson-normal is part of the class of infinitely divisible distributions.

We can write the underlying asset return as

$$
\begin{aligned}
R_t &= \kappa_t + \varepsilon_t, \text{ where} \\
\varepsilon_t &= \sigma_t \left(J_t - \vartheta\bar{\mu}\right)
\end{aligned}
$$

where J_t is a Poisson jump process with N_t jumps each with distribution $N\left(\bar{\mu}, \bar{\varphi}^2\right)$ and jump intensity ϑ. The conditional return variance equals $\left(1 + \vartheta\left(\bar{\mu}^2 + \bar{\varphi}^2\right)\right)\sigma_t^2$, where σ_t^2 is of the GARCH form. The log return mean κ_t is a function of σ_t^2 as well as the jump and risk premium parameters.

We can derive the conditional log MGF as

$$
\begin{aligned}
\Psi_t(u) &= \ln(E_{t-1}\left[\exp\left(-u\sigma_t\left(J_t - \vartheta\bar{\mu}\right)\right)\right]) \\
&= u\vartheta\bar{\mu}\sigma_t + \frac{1}{2}u^2\sigma_t^2 + \vartheta\left[\exp\left(-\bar{\mu}u\sigma_t + \frac{1}{2}\bar{\varphi}^2u^2\sigma_t^2\right) - 1\right]
\end{aligned}
$$

---

[16]Maheu and McCurdy (2004) consider a different discrete-time jump model but do not use it for option valuation.

The approach taken in Duan et al (2005) corresponds to fixing $\nu_t = \nu$ and setting

$$\kappa_t = r + \Psi_t(\nu) - \Psi_t(\nu - 1)$$

which in turn implies that the EMM is given by

$$\left.\frac{dQ}{dP}\right|F_t = \exp\left(-\nu t\overline{\varepsilon_t} - \nu\vartheta\overline{\mu}t\overline{\sigma_t} - \frac{1}{2}t\nu^2\overline{\sigma_t^2} + \vartheta t - \vartheta\sum_{i=1}^{t}\exp\left(-\overline{\mu}\nu_i\sigma_i + \frac{1}{2}\bar{\varphi}^2\nu_i^2\sigma_i^2\right)\right)$$

where $\overline{\varepsilon_t}$ and $\overline{\sigma_t^2}$ are the historical averages as above.

We can again show that the risk-neutral distribution is from the same family as the physical distribution

$$
\begin{aligned}
\Psi_t^{Q*}(u) &= \ln E_{t-1}^{Q}\left[\exp\left(-u\varepsilon_t^*\right)\right] \\
&= u\vartheta_t^*\overline{\mu}_t^*\sigma_t + \frac{1}{2}u^2\sigma_t^2 + \vartheta_t^*\left[\exp\left(-\overline{\mu}_t^*u\sigma_t + \frac{1}{2}\bar{\varphi}^2u^2\sigma_t^2\right) - 1\right]
\end{aligned}
$$

where

$$\vartheta_t^* = \vartheta\exp\left(-\overline{\mu}\nu\sigma_t + \frac{1}{2}\bar{\varphi}^2\nu^2\sigma_t^2\right) \text{ and } \overline{\mu}_t^* = \overline{\mu} - \bar{\varphi}^2\sigma_t\nu$$

Note that in this model the mapping between the risk-neutral and physical returns is

$$\varepsilon_t^* = \varepsilon_t + \Psi_t'(\nu) = \varepsilon_t + \sigma_t\left(\vartheta\overline{\mu} - \vartheta_t^*\overline{\mu}_t^*\right)$$

and the mapping between the physical and risk-neutral conditional variance is

$$\sigma_t^{*2} = \sigma_t^2 + \vartheta_t^*\sigma_t^2\left(\bar{\varphi}^2 + \overline{\mu}_t^{*2}\right)$$

6   Some continuous-time limits

In order to anchor our work in the continuous-time literature we now explore the links between some of the discrete-time models we have analyzed above and standard continuous-time models. We study three important cases: a homoskedastic model with normal innovations, a homoskedastic model with non-normal (inverse Gaussian) innovations, and a heteroskedastic model with normal innovations.

6.1 Homoskedastic normal returns

Consider the homoskedastic i.i.d. normal model for a given discrete-time interval Δ,

$$R_t = \ln(S_t) - \ln(S_{t-\Delta}) = \mu\Delta - \tfrac{1}{2}\sigma^2\Delta + \sigma\sqrt{\Delta}z_t \qquad\qquad z_t|F_{t-1} \sim N(0,1) \qquad (6.1)$$

and for simplicity also consider a constant risk-free rate. The EMM condition (2.4) is solved by choosing a constant $\nu = (\mu - r)/\sigma^2$, and the discrete-time risk-neutral dynamic is given by

$$\ln(S_t) - \ln(S_{t-\Delta}) = r\Delta - \tfrac{1}{2}\sigma^2\Delta + \sigma\sqrt{\Delta}z_t^* \qquad\qquad z_t^*|F_{t-1} \sim N(0,1) \qquad (6.2)$$

The continuous-time limit of this risk-neutral process is given by

$$d(\ln(S_t)) = \left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dz^*(t)$$

where $z^*(t)$ is a Wiener process under Q. This is the risk-neutral process in the Black-Scholes-Merton (BSM) model. In the diffusion limit the options are thus priced using the BSM formula.

Consider a European option with strike price K and $T - t = M\Delta$ days to maturity. The call price can be written as

$$C_{\Delta,t} = e^{-rM\Delta}S_t E_t^Q\left[e^{R_{t,M}}I[R_{t,M} > \ln(K/S_t)]\right] - e^{-rM\Delta}KP_t^Q[R_{t,M} > \ln(K/S_t)]$$

where $R_{t,M} = \ln(S_{t+M\Delta}) - \ln(S_t)$ and where $I[*]$ is the indicator function. Under the assumption of an i.i.d. normal risk-neutral process in (6.2) we can rewrite the call price as

$$C_{\Delta,t} = e^{-rM\Delta}S_t P_{1,t,\Delta} - e^{-rM\Delta}KP_{2,t,\Delta}$$

where

$$P_{1,t,\Delta} = e^{rM\Delta}\Phi\left(\frac{\ln(S_t/K) + \left(r + \tfrac{1}{2}\sigma^2\right)\Delta M}{\sigma\sqrt{\Delta M}}\right), \quad P_{2,t,\Delta} = \Phi\left(\frac{\ln(S_t/K) + \left(r - \tfrac{1}{2}\sigma^2\right)\Delta M}{\sigma\sqrt{\Delta M}}\right)$$

where Φ is the c.d.f. of the standard normal distribution.

Note therefore that for the i.i.d. normal discrete-time process, using the parameterization in (6.1), and given the choice of Radon-Nikodym derivative (and thus EMM), the option value is equal to the BSM price for any Δ. We will discuss this finding further in Section 7.

6.2 Homoskedastic inverse Gaussian returns

Consider now a homoskedastic version of the inverse Gaussian (IG) model in (5.1) written for a discrete-time interval Δ,

$$
\begin{aligned}
R_t &= r\Delta + \left(\zeta(\Delta) + \eta(\Delta)^{-1}\right)\sigma^2(\Delta) + \varepsilon_t \\
\varepsilon_t &= \eta(\Delta)y_t - \eta(\Delta)^{-1}\sigma^2(\Delta) \\
y_t &\sim IG\left(\frac{\sigma^2(\Delta)}{\eta^2(\Delta)}\right)
\end{aligned}
$$

As shown above for the heteroskedastic IG case, the risk neutral return distribution is in the same family as the historical model, and can be written as follows

$$
\begin{aligned}
R_t &= r\Delta + \left(\zeta^*(\Delta) + \eta^*(\Delta)^{-1}\right)\sigma^{*2}(\Delta) + \varepsilon_t^* \\
\varepsilon_t^* &= \eta^*(\Delta)y_t^* - \eta^*(\Delta)^{-1}\sigma^{*2}(\Delta) \\
y_t^* &\overset{Q}{\sim} IG\left(\frac{\sigma^{*2}(\Delta)}{\eta^{*2}(\Delta)}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta^*(\Delta) &= \frac{\eta(\Delta)}{1 + 2\nu(\Delta)\eta(\Delta)} \\
\sigma^{*2}(\Delta) &= \frac{\sigma^2(\Delta)}{(1 + 2\nu(\Delta)\eta(\Delta))^{3/2}} \\
\zeta^*(\Delta) &= \frac{1 - 2\eta^*(\Delta) - \sqrt{1 - 2\eta^*(\Delta)}}{\eta^*(\Delta)^2}
\end{aligned}
$$

and where $\nu(\Delta)$ solves (2.4) and is given by

$$
\nu(\Delta) = \frac{1}{2\eta(\Delta)}\left[\frac{(2 + \zeta(\Delta)^2\eta(\Delta)^3)^2}{4\zeta(\Delta)^2\eta(\Delta)^2} - 1\right]
$$

Consider a European option with strike price K and $T - t = M\Delta$ days to maturity. The call price can be written as

$$
C_{\Delta,t} = e^{-rM\Delta}S_t P_{1,t,\Delta} - e^{-rM\Delta}K P_{2,t,\Delta} \tag{6.3}
$$

The formulas for $P_{1,t,\Delta}$ and $P_{2,t,\Delta}$ can be computed using Fourier inversion of the risk-neutral log MGF of $\Psi_{t,M}^{Q*}(u)$

$$
P_{1,t,\Delta} = \frac{e^{rM\Delta}}{2} + \int_0^{+\infty} \text{Re} \left[\frac{\exp\left(\Psi_{t,M}^{Q*}(-1-iu) - iu \ln\left(\frac{K}{S_t}\right)\right)}{i\pi u} \right] du
$$

$$
P_{2,t,\Delta} = \frac{1}{2} + \int_0^{+\infty} \text{Re} \left[\frac{\exp\left(-iu \ln\left(\frac{K}{S_t}\right) + \Psi_{t,M}^{Q*}(-iu)\right)}{i\pi u} \right] du
$$

where

$$
\Psi_{t,M}^{Q*}(u) \equiv \ln\left(E_t^Q\left[\exp\left(uR_{t,M}\right)\right]\right) = -\left(r\Delta + \zeta(\Delta)\sigma^2(\Delta)\right)Mu + \frac{\left[\left(1 - \sqrt{1 + 2u\eta^*(\Delta)}\right)\sigma^{*2}(\Delta)M\right]}{\eta^*(\Delta)^2}
$$

Christoffersen, Heston and Jacobs (2006) show that in the heteroskedastic case, the stochastic volatility model in Heston (1993a) with perfectly correlated shocks can be obtained as a limit of the IG-GARCH model when Δ and $\eta(\Delta)$ go to zero.[17] This limit obtains when using a particular parameterization for the IG-GARCH model and the parameterization $\zeta(\Delta) = \lambda - \eta(\Delta)^{-1}$ for the return mean, where λ can be interpreted as the price of equity risk. As the homoskedastic IG model is a special case of the IG-GARCH model it will converge to the homoskedastic Heston (1993a) process which is simply the geometric Brownian motion underlying the Black-Scholes model. The continuous-time limit of the risk-neutral process is thus again given by

$$
d(\ln(S_t)) = \left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dz^*(t)
$$

Figure 2 illustrates the convergence of the homoskedastic IG option price in (6.3) to the BSM price when Δ goes to zero. In the figure we plot the ratio of the homoskedastic IG option price to the Black-Scholes price against the number of trading intervals per day. We use $r = 0$, $K = 100$, $S = 100$, $M\Delta = 180$. We let $\eta(\Delta) = \eta\Delta$, $\sigma^2(\Delta) = \sigma^2\Delta$, and set $\lambda\sigma^2 = .07$ to match a 7% equity risk premium. Return volatility is set to 10% per year ($\sigma^2 = .01$) in the top row and 20% in the bottom row ($\sigma^2 = .04$). The IG parameter η is set so as to generate a daily skewness of -1 in the left column and -0.5 in the right column. The figure shows that even for these relatively high levels of skewness the convergence of the skewed IG discrete-time option price to the Black-Scholes option price is quite rapid.

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[17]Christoffersen, Heston and Jacobs (2006) also show that an alternative pure jump limit can be obtained in the inverse Gaussian model.

6.3   Heteroskedastic normal returns

Consider the Heston and Nandi (2000) model

$$R_t = r\Delta + \lambda\sigma_t^2 + \sigma_t z_t \tag{6.4}$$

$$\sigma_{t+\Delta}^2 = \omega + \beta\sigma_t^2 + \alpha(z_t - \varrho\sigma_t)^2$$

Defining $v_{t+\Delta} = \sigma_{t+\Delta}^2/\Delta$, we have

$$v_{t+\Delta} = \omega_v + \beta v_t + \alpha_v\left(z_t - \varrho_v\sqrt{v_t}\right)^2 \tag{6.5}$$

with $\omega_v = \omega/\Delta$, $\alpha_v = \alpha/\Delta$ and $\varrho_v = \varrho\sqrt{\Delta}$. The conditional correlation is

$$Corr_{t-\Delta}\left(v_{t+\Delta}, R_t\right) = -\frac{sign(\varrho_v)\sqrt{2\varrho_v^2 v_t}}{\sqrt{1 + 2\varrho_v^2 v_t}}$$

so that the correlation goes to plus or minus one when the interval shrinks to zero. Using the parametrization $\alpha(\Delta) = \frac{1}{4}\varsigma^2\Delta^2$, $\beta(\Delta) = 0$, $\omega(\Delta) = (\kappa\theta - \frac{1}{4}\varsigma^2)\Delta^2$, and $\varrho(\Delta) = \frac{2}{\varsigma\Delta} - \frac{\kappa}{\varsigma}$, and following Foster and Nelson (1994), Heston and Nandi derive the diffusion limit for the physical process

$$\begin{aligned} d\ln(S_t) &= (r + \lambda v)dt + \sqrt{v}dz \\ dv &= \kappa(\theta - v)dt + \varsigma\sqrt{v}dz \end{aligned} \tag{6.6}$$

which corresponds to a special case of the stochastic volatility model in Heston (1993a) with perfectly correlated shocks to stock price and volatility.

The Heston-Nandi discrete-time option price is

$$C_{t,\Delta} = S_t P_{1,t,\Delta} - e^{-rM\Delta}KP_{2,t,\Delta}$$

where the formulas for $P_{1,t,\Delta}$ and $P_{2,t,\Delta}$, which rely on Fourier inversion, are provided in Heston and Nandi (2000).

Note that markets are complete in the limiting case with $\rho = -1$ because there is only one source of uncertainty. Below we analyze the more general case of a discrete-time two-shock stochastic volatility model and its continuous-time limit where $-1 < \rho < 1$, which implies that markets are incomplete even in continuous time.

Figure 3 shows the convergence of the Heston and Nandi (2000) discrete-time GARCH option price to the continuous-time SV option price in Heston (1993a). We plot the ratio of the Heston

and Nandi (2000) price to the Heston (1993a) price as the number of trading intervals until maturity gets large. We use $r = 0$, $K = 100$, $S = 100$, $M\Delta = 180$, $\kappa = 2$, and shock correlation $\rho = -1$. Return volatility is set to 10% per year ($v = \theta = .01$) in the top row and 20% in the bottom row ($v = \theta = .04$). The volatility of volatility parameter ς is set to 0.1 in the left column and 0.2 in the right column.

Figure 3 indicates that convergence is very fast, suggesting that the added incompleteness arising from discrete time is minimal. By comparison, convergence is slower in Figure 2 because of the conditional skewness in the discrete-time process. Note that following Heston and Nandi (2000), Figure 3 has trading intervals until maturity (180 days) on the horizontal axis whereas Figure 2 has trading intervals per day on the horizontal axis. Thus convergence is indeed extremely fast in Figure 3.

7   Stochastic Volatility Models

In this section, we first develop a discrete-time two-shock stochastic volatility model and derive its continuous-time limit. Subsequently we compare the risk neutralization for this model with the risk neutralization in the continuous-time SV model, and we discuss risk neutralization in the GARCH model as a special case of this approach. We also discuss the issue of market incompleteness and the resulting non-uniqueness of option prices, again by discussing similarities and differences between the continuous- and discrete-time setups.

7.1   A discrete-time stochastic volatility model

Popular continuous-time stochastic volatility models such as Heston (1993a) contain two (correlated) innovations, whereas the GARCH processes considered in this paper contain a single innovation. Nelson (1991) and Duan (1997) derive a continuous-time two-innovation stochastic volatility model as the limit of a GARCH model, but as noted by Corradi (2000) for instance, a given discrete-time model can have several continuous-time limits and vice versa.[18] As shown above, Heston and Nandi (2000) derive a limit to their proposed GARCH process that contains two perfectly correlated shocks. This limit amounts to a one-shock process, and is therefore intuitively similar to a GARCH process.

With this in mind, we now analyze the limits of a class of discrete-time stochastic volatility processes, which contain two (potentially correlated) shocks.[19] We derive the continuous-time limits for these processes, and then analyze the GARCH limit as a special case.

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[18]See also Nelson and Foster (1994), Foster and Nelson (1996), Nelson (1996) and Ritchken and Trevor (1999) for limit results.

[19]See Ghysels, Harvey and Renault (1995) for a review of discrete-time stochastic volatility models.

Consider the return and volatility dynamics

$$
\begin{aligned}
R_t &= \ln(S_t/S_{t-1}) = \mu_t + \sigma_t z_{1,t} \\
\sigma_{t+1}^2 &= f(\sigma_t^2, z_{2,t})
\end{aligned}
$$

where

$$
z_t \equiv (z_{1,t}, z_{2,t})' \sim N\left(\left((0,0)', \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \right)
$$

The log MGF is given by

$$
\Psi_t(u_1, u_2) = \log[E_{t-1}(\exp(-u_1 z_{1,t} - u_2 z_{2,t}))] = \frac{1}{2}\left[(u_1 + \rho u_2)^2 + \left(1 - \rho^2\right) u_2^2 \right]
$$

By analogy with the one-shock linear case (2.2), we define the following Radon-Nikodym derivative

$$
\frac{dQ}{dP}|F_t = \exp\left(-\sum_{i=1}^{t} (\nu_{1,i} z_{1,i} + \nu_{2,i} z_{2,i} + \Psi_i(\nu_{1,i}, \nu_{2,i})) \right) \tag{7.1}
$$

Using an approach similar to the one-shock case, one can show that the probability measure Q defined by the Radon-Nikodym derivative is an EMM if and only if

$$
\Psi_t(\nu_{1,t} - \sigma_t \nu_{2,t}) - \Psi_t(\nu_{1,t}, \nu_{2,t}) + \mu_t - r = \frac{1}{2}\sigma_t^2 - (\nu_{1,t} + \rho\nu_{2,t})\sigma_t + \mu_t - r = 0 \tag{7.2}
$$

This is one equation in two unknowns, namely $\nu_{1,t}$ and $\nu_{2,t}$. Thus the second shock provides a new source of non-uniqueness to be discussed further below.

The risk neutral log MGF is given by

$$
\begin{aligned}
E_{t-1}^Q[\exp(-u_1 z_{1,t} - u_2 z_{2,t})] &= E_{t-1}^P\left[\left(\frac{\frac{dQ}{dP}|F_t}{\frac{dQ}{dP}|F_{t-1}} \right) \exp(-u_1 z_{1,t} - u_2 z_{2,t}) \right] \\
&= \exp(\Psi_t(u_1 + \nu_{1,t}, u_2 + \nu_{2,t}) - \Psi_t(\nu_{1,t}, \nu_{2,t}))
\end{aligned}
$$

where

$$
z_t = (z_{1,t}, z_{2,t})' \stackrel{Q}{\sim} N\left(\left((-\nu_{1,t} - \rho\nu_{2,t}, -\nu_{1,t}\rho - \nu_{2,t})', \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \right) \tag{7.3}
$$

We now illustrate this risk-neutralization for a specific parametric example

$$
\begin{aligned}
R_t &= r + \lambda\sigma_t^2 - \frac{1}{2}\sigma_t^2 + \sigma_t z_{1,t} \\
\sigma_{t+1}^2 &= \omega + \beta\sigma_t^2 + \alpha(z_{2,t} - \varrho\sigma_t)^2
\end{aligned} \tag{7.4}
$$

The dynamic in (7.4) can be thought of as a stochastic volatility (two-shock) generalization of the GARCH dynamic in Heston and Nandi (2000). According to (7.3) the risk-neutral model is given by

$$
\begin{aligned}
R_t &= r - \frac{1}{2}\sigma_t^2 + \sigma_t z_{1,t}^* \\
\sigma_{t+1}^2 &= \omega + \beta\sigma_t^2 + \alpha(z_{2,t}^* - \nu_{1,t}\rho - \nu_{2,t} - \varrho\sigma_t)^2
\end{aligned}
\tag{7.5}
$$

where

$$
z_t^* = \begin{pmatrix} z_{1,t}^* = z_{1,t} + \nu_{1,t} + \rho\nu_{2,t} \\ z_{2,t}^* = z_{2,t} + \nu_{1,t}\rho + \nu_{2,t} \end{pmatrix} \overset{Q}{\sim} N\left(\left((0,0)', \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \right)
$$

In the one-shock GARCH case above, we could simply solve (2.4) by choosing the scalar ν_t as a function of the GARCH parameters. Determining $\nu_{1,t}$ and $\nu_{2,t}$ in a model with two innovations is somewhat more complex, but the intuition underlying the procedure is critical to understanding the link with the continuous-time literature. From (7.2) and (7.4) we have $\nu_{1,t} + \nu_{2,t}\rho = \lambda\sigma_t$. We then note that if we want to preserve the affine structure in (7.5) we need $\nu_{2,t} = \nu_2\sigma_t$, which yields the risk neutral dynamic

$$
\begin{aligned}
R_t &= r - \frac{1}{2}\sigma_t^2 + \sigma_t z_{1,t}^* \\
\sigma_{t+1}^2 &= \omega + \beta\sigma_t^2 + \alpha(z_{2,t}^* - \varrho^*\sigma_t)^2
\end{aligned}
\tag{7.6}
$$

with $\varrho^* = \varrho + \nu_2(1 - \rho^2) + \lambda\rho$. The condition on the price of risk needed to preserve the affine structure is similar to the one usually used in the Heston (1993a) model. Note that conditional on the assumption regarding the price of volatility risk, Proposition 2 can be generalized to address existence and uniqueness of a solution to (7.2).

Note that while λ, which is the price of equity risk, can be estimated from returns, ν_2, which arises from the new separate volatility shock, is not identified from the return on the underlying asset only. It must be estimated using returns as well as option prices. This is of course also the case in continuous-time SV models. The analysis is therefore very similar to the one usually employed in continuous time.

Using an approach similar to that taken in Heston and Nandi (2000), option valuation in this discrete-time SV model can be done via Fourier inversion of the conditional characteristic function.

7.2 A diffusion limit of the discrete-time stochastic volatility model

We first write the discrete-time stochastic volatility model as

$$R_t = r\Delta + \lambda\sigma_t^2 - \frac{1}{2}\sigma_t^2 + \sigma_t z_{1,t} \tag{7.7}$$

$$\sigma_{t+\Delta}^2 = \omega + \beta\sigma_t^2 + \alpha(z_{2,t} - \varrho\sigma_t)^2 \tag{7.8}$$

Reparametrizing $v_{t+\Delta} = \sigma_{t+\Delta}^2/\Delta$, we have

$$v_{t+\Delta} = \omega_v + \beta v_t + \alpha_v \left(z_{2,t} - \varrho_v\sqrt{v_t}\right)^2 \tag{7.9}$$

with $\omega_v = \omega/\Delta$, $\alpha_v = \alpha/\Delta$ and $\varrho_v = \varrho\sqrt{\Delta}$.

Following Heston and Nandi (2000) we use the parametrization $\alpha(\Delta) = \frac{1}{4}\varsigma^2\Delta^2$, $\beta(\Delta) = 0$, $\omega(\Delta) = (\kappa\theta - \frac{1}{4}\varsigma^2)\Delta^2$, and $\varrho(\Delta) = \frac{2}{\varsigma\Delta} - \frac{\kappa}{\varsigma}$. As $\Delta \to 0$ the dynamic in (7.7) and (7.9) converges to

$$d\ln(S_t) = (r + \lambda v_t - \frac{1}{2}v_t)dt + \sqrt{v_t}dz_1 \tag{7.10}$$

$$dv_t = \kappa(\theta - v_t)dt + \varsigma\sqrt{v_t}dz_2$$

where z_1 and z_2 are two Wiener processes such that $dz_1 dz_2 = -\rho dt$. Note that the discrete-time conditional correlation is given by

$$corr_{t-\Delta}\left(v_{t+\Delta}, R_t\right) = -\frac{\rho sign(\varrho_v)\sqrt{2\varrho_v^2 v_t}}{\sqrt{1 + 2\varrho_v^2 v_t}}$$

As $\Delta \to 0$, the variance asymmetry parameter $\varrho_v(\Delta)$ approaches positive or negative infinity, and therefore the correlation approaches ρ or $-\rho$ in the limit. Also, as $\Delta \to 0$, the risk neutral discrete-time stochastic volatility model (7.6) converges to the following dynamic

$$d\ln(S_t) = (r - \frac{1}{2}v_t)dt + \sqrt{v_t}dz_1^* \tag{7.11}$$

$$dv_t = [\kappa(\theta - v_t) + \varsigma(\nu_2(1 - \rho^2) + \lambda\rho)v_t]dt + \varsigma\sqrt{v_t}dz_2^*$$

where z_1^* and z_2^* are two Wiener processes such that $dz_1^* dz_2^* = -\rho dt$.

7.3 The relationship with the continuous-time affine SV model

Both (7.10) and (7.11) are square root stochastic volatility models of the type proposed by Heston (1993a). We now link our discrete-time stochastic volatility model and its risk-neutralization to

the conventional risk-neutralization in the Heston (1993a) model. Assume for simplicity that the parameterization of the conditional mean dynamic under the physical measure is given by (7.10). Heston (1993a) proposes the following risk neutralization[20]

$$
\begin{aligned}
d\ln(S_t) &= (r - \frac{1}{2}v_t)dt + \sqrt{v_t}dz_1^* \\
dv_t &= [\kappa(\theta - v_t) - \varsigma\chi^* v_t]dt + \varsigma\sqrt{v_t}dz_2^*
\end{aligned}
\tag{7.12}
$$

where z_1^* and z_2^* are two Wiener process under the risk neutral probability Q and

$$
\begin{aligned}
dz_1^* &= dz_1 + \left(\lambda - \frac{1}{2}\right)\sqrt{v_t}dt \\
dz_2^* &= dz_2 + \chi^*\sqrt{v_t}dt
\end{aligned}
\tag{7.13}
$$

In the discrete-time stochastic volatility model, the parameter λ in (7.4) captures the price of equity risk, and ν_2 captures the price of volatility risk. In the Heston model, the price of equity risk λ plays the same role as in the discrete-time model, and we have also a price of volatility risk χ^* which ensures the affine structure of the risk-neutral process. Comparing (7.12) and (7.11), we find

$$
\chi^* = \nu_2(1 - \rho^2) + \lambda\rho.
\tag{7.14}
$$

which amounts to the assumption on the price of risk used in Pan (2002). Note that for $\rho = 0$, the continuous-time price of volatility risk χ^* is not related to λ, but is simply equal to the discrete-time price of volatility risk ν_2. Moreover, this mapping between the price of volatility risk in discrete-time and continuous-time stochastic volatility models also provides insight into the relationship between the discrete-time GARCH model and the available continuous-time literature. While the GARCH model contains a single innovation, it can usefully be thought of as a special case of the two-shock discrete-time stochastic volatility model in (7.5), for $\rho = 1$ (or $\rho = -1$). In this case, from (7.14), $\chi^* = \lambda$ (or $-\lambda$). Because the GARCH model contains a single shock, the specification of the equity risk premium λ does double duty: it also implicitly defines the price of volatility risk, which is perfectly correlated with the price of equity risk by design. In other words, the GARCH return dynamic implicitly makes an assumption about the volatility risk premium. The parameter governing the equity risk premium also determines the volatility risk premium. Strictly speaking therefore, in the case of the GARCH model the only assumption we make in our approach is on the form of the Radon-Nikodym derivative. All other assumptions needed for risk-neutral valuation are implicit in the specification of the

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[20]Notice that for ease of interpretation, in our notation the price of volatility risk χ^* has been rescaled by $1/\varsigma$ compared to the notation in Heston (1993a).

return dynamic. Put differently, some important assumptions on the equilibrium supporting the valuation problem are implicitly incorporated in the risk premium assumption for the return dynamic.

7.4   Stochastic Volatility and GARCH

The discussion above indicates that while it is useful to distinguish between one-shock and two-shock models, our analysis of discrete-time GARCH option valuation models is very similar to the analysis of continuous-time SV option valuation models. Most existing papers on option pricing in discrete time assume normally distributed returns and, in the words of Rubinstein (1976), "complete" the markets by assuming a representative agent with certain preferences, such as for instance constant relative risk aversion.[21] Our approach, much like the one used in the continuous-time stochastic volatility literature, is to let the researcher specify an empirically realistic return dynamic for the underlying asset, and subsequently provide an equivalent martingale measure that enables option pricing using a no-arbitrage argument. Proposition 1 provides the form of the EMM and Proposition 5 provides the no-arbitrage option pricing result. Whereas the assumption on the representative agent's utility function "completes" the market in the standard normal discrete-time setting, the Radon-Nikodym derivative "completes" the market in our setup. Conditional on the choice of Radon-Nikodym derivative which is linear in the return innovation, our approach provides a unique EMM.

The only difference between GARCH option valuation and option valuation with stochastic volatility is that GARCH models can be viewed as special cases of discrete-time stochastic volatility models. In the GARCH model, one parameter determines the volatility risk premium as well as the equity risk premium, and therefore the volatility risk premium is implicitly specified by the GARCH dynamic. This is consistent with the interpretation of the GARCH model as a one-shock model with perfectly correlated equity and volatility innovations.[22]

Section 3 illustrates that it is possible to generalize the EMM specification, although in most cases it is not straightforward to obtain analytical results. We therefore limit our discussion to the case of the quadratic EMM with normal innovations in Section 3.3, which contains the linear EMM as a special case. This indicates that the uniqueness result obtained for the GARCH model discussed above is due to the choice of the linear EMM. In the more general quadratic case, we obtain an infinite number of valid EMMs, as illustrated in Figure 1.

---

[21]See for example Rubinstein (1976), Brennan (1979), and Duan (1995).

[22]While it could be argued that this structure limits the usefulness of the GARCH model, one has to keep in mind that this structure is exactly what makes the GARCH model econometrically tractable. Indeed, the success of the GARCH model in modeling returns, and its growing popularity in modeling options, are precisely due to the fact that despite its simple structure it provides a very good fit.

8   Conclusion

This paper provides valuation results for contingent claims in a discrete-time infinite state space setup. Most of our analysis focuses on a class of Radon-Nikodym derivatives for which the risk neutral return dynamic is the same as the physical dynamic for a wide class of processes, but with a different parameterization which we are able to characterize completely. We also discuss more general choices of Radon-Nikodym derivatives. Our valuation argument applies to a large class of conditionally normal and non-normal stock returns with flexible time-varying mean and volatility, as well as a potentially time-varying price of risk. This setup generalizes the result in Duan (1995) in the sense that we do not restrict the returns to be conditionally normal, nor do we restrict the price of risk to be constant.

Our results apply to some of the most widely used discrete-time processes in finance, such as GARCH processes. We also apply our approach to the analysis of discrete-time processes with multiple innovations, such as discrete-time stochastic volatility processes. To provide intuition for our findings, we extensively discuss the relationship between our results and existing results for continuous-time stochastic volatility models, which can be derived as limits of our discrete-time dynamics.

Our results suggest a number of interesting avenues for further research. First, an extensive empirical comparison of option valuation with non-normal and heteroskedastic innovations should prove interesting. Combining non-normality and heteroskedasticity attempts to correct the biases associated with the conditionally normal GARCH model. These biases are similar to those displayed by the Heston (1993a) model, which the continuous-time literature has sought to remedy by adding (potentially correlated) jumps in returns and volatility.[23] A comparison with these models may prove valuable. Second, it is well-known that the risk-neutralization of existing models is not satisfactory from an empirical perspective.[24] The implications of alternative Radon-Nikodym derivatives for the option valuation models' empirical performance therefore ought to be studied. A comparison between linear and quadratic EMMs for normal innovations may provide a valuable starting point. Third, while we advocate separating the valuation issue and the general equilibrium setup that supports it, the general equilibrium foundations of our results are of course very important. It may prove possible to characterize the equilibrium setup that gives rise to the risk neutralization proposed for some of the processes considered in this paper. However, this is by no means a trivial problem, and it is left for future work.

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[23] See for example Bakshi, Cao and Chen (1997), Bates (2000), Broadie, Chernov and Johannes (2007), Carr and Wu (2004), Eraker, Johannes and Polson (2003), Eraker (2004), Huang and Wu (2004) and Pan (2002).

[24] See for example Broadie, Chernov and Johannes (2007).

9   Appendix

9.1   Proof of Proposition 2.

Define $f(\nu) = \Psi(\nu) - \Psi(\nu - 1)$. Existence is obtained if $f(\nu)$ can take any real value. Uniqueness is demonstrated if $f(\nu) = E[R - r] \equiv \pi$ has a unique solution for any given value of π. By assumption, Ψ tends to infinity at the boundaries of its domain, therefore $\Psi(u_1) = +\infty$ and $\Psi(u_2) = +\infty$. Ψ is also continuous because it is twice differentiable on its domain. The domain of $f(\nu)$ is $(u_1 + 1, u_2)$. Since Ψ is continuous $f(.)$ is also continuous. We get

$$
\begin{aligned}
f(u_1 + 1) &= \Psi(u_1 + 1) - \Psi(u_1) = -\Psi(u_1) = -\infty \\
f(u_2) &= \Psi(u_2) - \Psi(u_2 - 1) = \Psi(u_2) = +\infty
\end{aligned}
$$

since $\Psi(u_1) = +\infty$ and $\Psi(u_2) = +\infty$. Hence $f(.)$ is continuous and can attain $-\infty$ or $+\infty$. Thus there exists a value ν in the domain of the continuous function $f(.)$ such that $f(\nu) = \pi$ for any value $\pi \in (-\infty, +\infty)$. Furthermore, we have that $f'(u) = \Psi'(u) - \Psi'(u - 1)$. Convexity of Ψ implies that $\Psi'(.)$ is increasing. Thus, if $f'(u) = \Psi'(u) - \Psi'(u - 1) > 0$, then $f(.)$ is increasing. Therefore, $f(.)$ is increasing and continuous, which implies that $f(.)$ is a bijection, and uniqueness follows.

9.2   Proof of Lemma 4.

For a self financing strategy we have

$$
\begin{aligned}
G_{t+1} = V_{t+1} &= \eta_{t+1} S_{t+1} + \delta_{t+1} C_{t+1} + \psi_{t+1} B_{t+1} \\
&= \eta_t S_{t+1} + \delta_t C_{t+1} + \psi_t B_{t+1}
\end{aligned}
$$

We also have

$$
G_t = \sum_{i=0}^{t-1} \eta_i (S_{i+1} - S_i) + \sum_{i=0}^{t-1} \delta_i (C_{i+1} - C_i) + \sum_{i=0}^{t-1} \psi_i (B_{i+1} - B_i).
$$

It follows that

$$
G_{t+1} - G_t = \eta_t (S_{t+1} - S_t) + \delta_t (C_{t+1} - C_t) + \psi_t (B_{t+1} - B_t)
$$

We can trivially also write

$$G_{t+1}^B - G_t^B = G_{t+1}^B - G_t^B + \underbrace{\left(\frac{G_{t+1}}{B_t} - \frac{G_{t+1}}{B_t} \right)}_{=0}$$

This implies that

$$G_{t+1}^B - G_t^B = (\eta_t S_{t+1} + \delta_t C_{t+1} + \psi_t B_{t+1}) \left(\frac{1}{B_{t+1}} - \frac{1}{B_t} \right)$$
$$+ \frac{1}{B_t} \left(\eta_t (S_{t+1} - S_t) + \delta_t (C_{t+1} - C_t) + \psi_t (B_{t+1} - B_t) \right).$$

$$= \eta_t \left[S_{t+1} \left(\frac{1}{B_{t+1}} - \frac{1}{B_t} \right) + \frac{1}{B_t} (S_{t+1} - S_t) \right]$$
$$+ \delta_t \left[C_{t+1} \left(\frac{1}{B_{t+1}} - \frac{1}{B_t} \right) + \frac{1}{B_t} (C_{t+1} - C_t) \right]$$
$$+ \underbrace{ \psi_t B_{t+1} \left(\frac{1}{B_{t+1}} - \frac{1}{B_t} \right) + \frac{1}{B_t} \psi_t (B_{t+1} - B_t) }_{=0}$$

Then

$$G_{t+1}^B - G_t^B = \eta_t \left[S_{t+1} \left(\frac{1}{B_{t+1}} - \frac{1}{B_t} \right) + \frac{1}{B_t} (S_{t+1} - S_t) \right] + \delta_t \left[C_{t+1} \left(\frac{1}{B_{t+1}} - \frac{1}{B_t} \right) + \frac{1}{B_t} (C_{t+1} - C_t) \right]$$

$$= \eta_t (S_{t+1}^B - S_t^B) + \delta_t (C_{t+1}^B - C_t^B) + \left(\eta_t \frac{S_{t+1}}{B_t} - \eta_t \frac{S_{t+1}}{B_t} \right) + \left(\delta_t \frac{C_{t+1}}{B_t} - \delta_t \frac{C_{t+1}}{B_t} \right)$$

and therefore

$$G_{t+1}^B - G_t^B = \eta_t (S_{t+1}^B - S_t^B) + \delta_t (C_{t+1}^B - C_t^B). \qquad \forall t = 1, ..., T-1$$

Because $G_0 = G_0^B = 0$ the discounted gain can be written as the sum of past changes

$$G_t^B = \sum_{i=0}^{t-1} (G_{i+1}^B - G_i^B) \; \forall t = 1, ..., T.$$

Therefore the discounted gain can be written

$$G_t^B = \sum_{i=0}^{t-1} \eta_i (S_{i+1}^B - S_i^B) + \sum_{i=0}^{t-1} \delta_i (C_{i+1}^B - C_i^B)$$

and the proof is complete.

9.3 Proof of Proposition 3.

From Lukacs (1970), page 119, we have the Kolmogorov canonical representation of the log-moment generating function of an infinitely divisible distribution function. This result stipulates that a function Ψ is the log-moment generating function of an infinitely divisible distribution with finite second moment if, and only if, it can be written in the form

$$\Psi(u) = -uc + \int_{-\infty}^{+\infty} \left(e^{-ux} - 1 + ux\right) \frac{dK(x)}{x^2}$$

where c is a real constant while $K(u)$ is a nondecreasing and bounded function such that $K(-\infty) = 0$. Applying this theorem gives the following form for $\Psi_t(u)$,

$$\Psi_t(u) = -uc_{t-1} + \int_{-\infty}^{+\infty} \left(e^{-ux} - 1 + ux\right) \frac{dK_{t-1}(x)}{x^2} \tag{9.1}$$

where c_{t-1} is a random variable known at $t-1$, and $K_{t-1}(x)$ is a function known at $t-1$, which is nondecreasing and bounded so that $K_{t-1}(-\infty) = 0$. Using relation (2.9) and the characterization (9.1) we can write $\Psi_t^{Q*}(u)$ as

$$\Psi_t^{Q*}(u) = \int_{-\infty}^{+\infty} \left(e^{-ux} - 1 + ux\right) \frac{dK_{t-1}^*(x)}{x^2}$$

where

$$K_{t-1}^*(x) = \int_{-\infty}^{x} e^{-\nu_t y} dK_{t-1}(y)$$

This implies that

$$K_{t-1}^*(-\infty) = 0$$

$K_{t-1}^*(x)$ is obviously non-decreasing since $K_{t-1}(x)$ is non-decreasing, $K_{t-1}^*(\infty) < \infty$, because $K_{t-1}(\infty) < \infty$, and $e^{-\nu_t y}$ is a decreasing function of y which converge to 0. Recall that ν_t is the generalized price of risk, which is positive and known at time $t-1$.

In conclusion we have constructed a constant c_{t-1}^* ($= 0$) and a non-decreasing bounded function $K_{t-1}^*(x)$, with $K_{t-1}^*(-\infty) = 0$, such that

$$\Psi_t^{Q*}(u) = -uc_{t-1}^* + \int_{-\infty}^{+\infty} \left(e^{-ux} - 1 + ux\right) \frac{dK_{t-1}^*(x)}{x^2}.$$

Hence, according to the Kolmogorov canonical representation, the conditional distribution of ε_t^* is infinitely divisible.

References

[1] Amin, K. and V. Ng (1993), ARCH Processes and Option Valuation, Manuscript, University of Michigan.

[2] Bakshi, C., Cao, C. and Z. Chen (1997), Empirical Performance of Alternative Option Pricing Models, Journal of Finance, 52, 2003-2049.

[3] Barone-Adesi, G., R. Engle, and L. Mancini (2008), A Garch Option Pricing Model in Incomplete Markets, Review of Financial Studies, 21, 1223-1258.

[4] Bates, D. (1996), Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, Review of Financial Studies, 9, 69-107.

[5] Bates, D. (2000), Post-'87 Crash Fears in the S&P 500 Futures Option Market, Journal of Econometrics, 94, 181-238.

[6] Bates, D. (2003), Empirical Option Pricing: a Retrospection, Journal of Econometrics, 116, 387-404.

[7] Bernardo, A., and O. Ledoit (2000), Gain, Loss, and Asset Pricing, Journal of Political Economy, 108, 144-172.

[8] Bick, A. (1990), On Viable Diffusion Price Processes of the Market Portfolio, Journal of Finance, 45, 673-689.

[9] Black, F., and M. Scholes (1973), The Pricing of Options and Corporate Liabilities, Journal of Political Economy, 81, 637-659.

[10] Bollerslev, T. (1986), Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics, 31, 307-327.

[11] Bollerslev, T., Chou, R. and K. Kroner (1992), ARCH Modelling in Finance: A Review of the Theory and Empirical Evidence, Journal of Econometrics, 52, 5-59.

[12] Bollerslev, T., M. Gibbons, and H. Zhou (2005), Dynamic Estimation of Volatility Risk Premia and Investor Risk Aversion from Option-Implied and Realized Volatilities, Manuscript, Duke University.

[13] Bollerslev, T. and H.O. Mikkelsen (1996), Long-Term Equity AnticiPation Securities and Stock Market Volatility Dynamics, Journal of Econometrics, 92, 75-99.

[14] Bondarenko, O., and I. Longarela (2004), Benchmark Good Deal Bounds: an Application to Stochastic Volatility Models of Option Pricing, Working Paper, Stockholm School of Economics.

[15] Brennan, M. (1979), The Pricing of Contingent Claims in Discrete-Time Models, Journal of Finance, 34, 53-68.

[16] Broadie, M., Chernov, M. and M. Johannes (2007), Model Specification and Risk Premia: Evidence from S&P500 Futures Options Market, Journal of Finance, 62, 1453-1490.

[17] Buhlmann, H., F. Delbaen, P. Embrechts, and A. Shiryaev (1996), No-Arbitrage, Change of Measure and Conditional Esscher Transforms, CWI Quarterly, 9, 291-317.

[18] Buhlmann, H., Delbaen, F., Embrechts, P., and A. Shiryaev (1998), On Esscher Transforms in Discrete Finance Models, ASTIN Bulletin, 28, 171–186.

[19] Camara, A. (2003), A Generalization of the Brennan-Rubinstein Approach for the Pricing of Derivatives, Journal of Finance, 58, 805-819.

[20] Carr, P. and L. Wu (2004), Time-Changed Levy Processes and Option Pricing, Journal of Financial Economics, 17, 113–141.

[21] Carr, P. and L. Wu (2007), Variance Risk Premia, forthcoming in the Review of Financial Studies.

[22] Christoffersen, P., S. Heston, and K. Jacobs (2006), Option Valuation with Conditional Skewness, Journal of Econometrics, 131, 253-284.

[23] Christoffersen, P. and K. Jacobs (2004), Which GARCH Model for Option Valuation? Management Science, 50, 1204-1221.

[24] Cochrane, J., and J. Saa-Requejo (2000), Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets, Journal of Political Economy, 108, 79-119.

[25] Constantinides, G. M. and S. Perrakis (2002), Stochastic Dominance Bounds on Derivative prices in a Multiperiod Economy with Proportional Transaction Costs, Journal of Economic Dynamics and Control, 26, 1323-1352.

[26] Constantinides, G. M. and S. Perrakis (2007), Stochastic Dominance Bounds on American Option Prices in Markets with Frictions, Review of Finance 11, 71-115.

[27] Constantinides, G., Jackwerth, J. and S. Perrakis (2008), Mispricing of S&P 500 Index Options, forthcoming in the Review of Financial Studies.

[28] Corradi, V. (2000), Reconsidering the Continuous-Time Limit of the GARCH(1,1) Process, Journal of Econometrics, 96, 145-153.

[29] Cox, J. and S. Ross (1976), The Valuation of Options for Alternative Stochastic Processes, Journal of Financial Economics, 3, 145-166.

[30] Cox, J., Ross, S. and M. Rubinstein (1979), Option Pricing: A Simplified Approach, Journal of Financial Economics, 7, 229-264.

[31] Dai, Q., Le, A., and K. Singleton (2006), Discrete-Time Dynamic Term Structure Models with Generalized Market Prices of Risk, Manuscript, Stanford University.

[32] Diebold, F. and J. Lopez (1995), Modeling Volatility Dynamics, in Macroeconometrics: Developments, Tensions and Prospects, Kevin Hoover (ed.). Boston: Kluwer Academic Press.

[33] Duan, J.-C. (1995), The GARCH Option Pricing Model, Mathematical Finance, 5, 13-32.

[34] Duan, J.-C. (1997), Augmented GARCH(p,q) Process and Its Diffusion Limit, Journal of Econometrics, 79, 97-127.

[35] Duan, J.-C. (1999), Conditionally Fat-Tailed Distributions and the Volatility Smile in Options, Manuscript, University of Toronto.

[36] Duan, J.-C., Ritchken, P. and Z. Sun (2005), Jump Starting GARCH: Pricing and Hedging Options with Jumps in Returns and Volatilities, Working Paper, Rotman School, University of Toronto.

[37] Engle, R. (1982), Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK Inflation, Econometrica, 50, 987-1008.

[38] Engle, R., D. M. Lilien and R.P. Robins (1987), Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Model, Econometrica, 55, 391-407.

[39] Engle, R. and V. Ng (1993), Measuring and Testing the Impact of News on Volatility, Journal of Finance, 48, 1749-1778.

[40] Eraker, B. (2004), Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices, Journal of Finance, 59, 1367-1403.

[41] Eraker, B., M. Johannes and N. Polson (2003), The Impact of Jumps in Volatility and Returns, Journal of Finance, 58, 1269-1300.

[42] Feller, W. (1968), An Introduction to Probability Theory and its Applications. John Wiley & Sons.

[43] Foster, D. P. and D.B. Nelson (1996), Continuous Record Asymptotics for Rolling Sample Variance Estimators, Econometrica, 64, 139-174.

[44] French, K., G. W. Schwert, and R. Stambaugh (1987), Expected Stock Returns and Volatility, Journal of Financial Economics, 19, 3-30.

[45] Garcia, R. and E. Renault (1998), A Note on Hedging in ARCH-type Option Pricing Models, Mathematical Finance, 8, 153-161.

[46] Gerber, H., and E. Shiu (1994), Option Pricing By Esscher Transforms, Transactions of the Society of Actuaries, 46, 99-191.

[47] Glosten, L. R., R. Jagannathan, and D. Runkle (1993), On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, Journal of Finance, 48, 1779-1801.

[48] Ghysels, E., A. Harvey, and E. Renault (1995), Stochastic Volatility, Handbook of Statistics 14, Statistical Methods in Finance, G.S. Maddala and C.R. Rao (eds.), North Holland. Amsterdam.

[49] Gourieroux, C. and A. Monfort (2007), Econometric Specification of Stochastic Discount Factor Models, Journal of Econometrics, 136, 509-530.

[50] Harrison, M. and D. Kreps (1979), Martingales and Arbitrage in Multi-period Securities Markets, Journal of Economic Theory, 20, 381-408.

[51] Harrison, M., and S. Pliska (1981), Martingales and Stochastic Integrals in the Theory of Continuous Trading, Stochastic Processes and their Applications, 11, 215–260.

[52] He, H. and H. Leland (1993), On Equilibrium Asset Price Processes, Review of Financial Studies, 6, 593-617.

[53] Heston, S. (1993a), A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, Review of Financial Studies, 6, 327-343.

[54] Heston, S. (1993b), Invisible Parameters in Option Prices, Journal of Finance, 48, 933-947.

[55] Heston, S. (2004), Option Valuation with Infinitely Divisible Distributions, Quantitative Finance, 4, 515-524.

[56] Heston, S. and S. Nandi (2000), A Closed-Form GARCH Option Pricing Model, Review of Financial Studies, 13, 585-626.

[57] Huang, J.-Z. and L. Wu (2004), Specification Analysis of Option Pricing Models Based on Time-Changed Levy Processes, Journal of Finance, 59, 1405–1439.

[58] Jacod, J. and A. Shiryaev (1998), Local Martingales and the Fundamental Asset Pricing Theorems in the Discrete-time Case, Finance and Stochastics, 259-273.

[59] Karatzas, I., and S. Shreve (1998), Methods of Mathematical Finance, Springer-Verlag.

[60] Levy, H. (1985), Upper and Lower Bounds of Put and Call Option Value: Stochastic Dominance Approach, The Journal of Finance, 40, 1197-1217.

[61] Lukacs, E. (1970), Characteristic Functions, Griffin. London

[62] Madan, D. B., and E. Seneta (1990), The Variance Gamma Model from Share Market Returns, Journal of Business, 63, 511-524.

[63] Maheu, J., and T. McCurdy (2004), News Arrival, Jump Dynamics and Volatility Components for Individual Stock Returns, Journal of Finance, 59, 755–793.

[64] Nelson, D. B. (1991), Conditional Heteroskedasticity in Asset Returns: A New Approach, Econometrica, 59, 347-370.

[65] Nelson, D. B. (1996), Asymptotically Optimal Smoothing with Arch Models, Econometrica, 64, 561-573.

[66] Nelson, D. B. and D. P. Foster (1994), Asymptotic Filtering Theory for Univariate Arch Models, Econometrica, 62, 1-41.

[67] Oancea, I. and S. Perrakis (2007), Stochastic Dominance and Option Pricing in Discrete and Continuous Time: an Alternative Paradigm, Working Paper, Concordia University.

[68] Pan, J. (2002), The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study, Journal of Financial Economics, 63, 3–50.

[69] Perrakis, S. (1986), Option Bounds in Discrete Time: Extensions and the Pricing of the American Put, Journal of Business, 59, 119-141.

[70] Perrakis, S. and P. Ryan (1984), Option Bounds in Discrete Time, Journal of Finance, 39, 519-525.

[71] Ritchken, P.(1985), On Option Pricing Bounds, The Journal of Finance, 40, 1219-1233.

[72] Ritchken, P. and S. Kuo (1988), Option Bounds with Finite Revision Opportunities, Journal of Finance, 43, 301-308.

[73] Ritchken, P. and R. Trevor (1999), Option Pricing Under GARCH and Stochastic Volatility, Journal of Finance, 54, 377-402.

[74] Rubinstein, M. (1976), The Valuation of Uncertain Income Streams and the Pricing of Options, Bell Journal of Economics, 7, 407-425.

[75] Satchell, S., and A. Timmermann (1996), Option Pricing with GARCH and Systematic Consumption Risk, Derivatives Use, Trading & Regulation, Part II, 1, 353-67.

[76] Schroder, M. (2004), Risk Neutral Parameter Shifts and Derivatives Pricing in Discrete Time, Journal of Finance, 59, 2375-2402.

[77] Schwert, G.W. (1989), Why Does Stock Market Volatility Change over Time? Journal of Finance, 44, 1115-1153.

[78] Shiryaev, A. (1999), Essentials of Stochastic Finance. Facts, Models, Theory, World Scientific.

[79] Siu, T.K., H. Tong, and H. Yang (2004), On Pricing Derivatives under GARCH Models: A Dynamic Gerber-Shiu Approach, North American Actuarial Journal, 8, 17-31.

Figure 1: Option Prices from Linear and Quadratic EMMs.
One-Month-to-Maturity, At-the-Money Call.
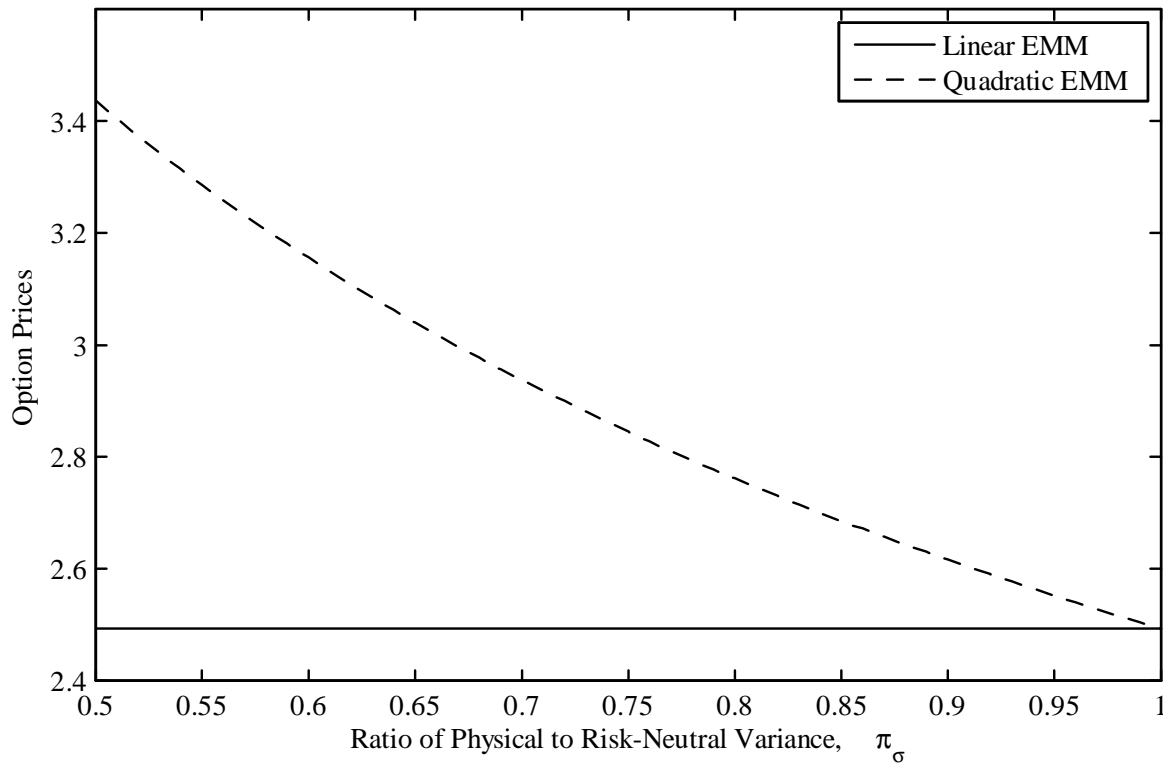
Notes to Figure: We use the linear and quadratic EMMs to compute the price of a one-month to maturity, at-the-money call option with an underlying asset price of 100. We assume a risk-free rate of 5%, an underlying asset mean return of 10% and a physical asset volatility of 20% per year. In the quadratic EMM we let the ratio of the physical to risk-neutral variance, π_σ vary from 0.5 to 1.

Figure 2: Convergence of Homoskedastic Inverse Gaussian to Black-Scholes Option Price
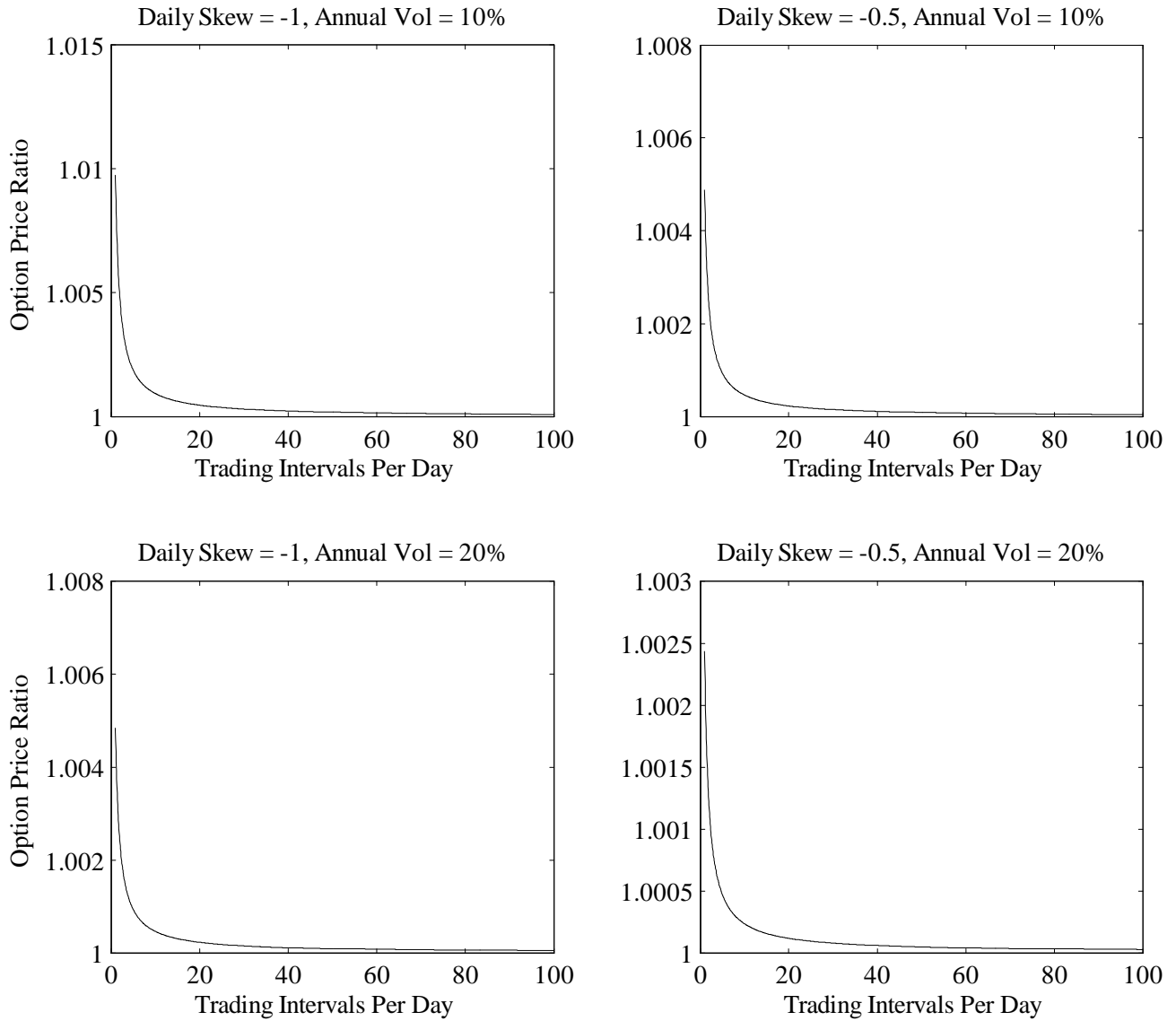
Notes to Figure: We plot the ratio of the homoskedastic IG option price to the Black-Scholes price as the number of trading intervals per day gets large. We use $r = 0$, $K = 100$, $S = 100$, $M\Delta = 180$. We let $\eta(\Delta) = \eta\Delta$, $\sigma^2(\Delta) = \sigma^2\Delta$, and set $\lambda\sigma^2 = .07$ to match a 7% equity risk premium. Return volatility is set to 10% per year ($\sigma^2 = .01$) in the top row and 20% in the bottom row ($\sigma^2 = .04$). The IG parameter η is set so as to generate a daily skewness of -1 in the left column and -0.5 in the right column.

Figure 3: Convergence of GARCH to Stochastic Volatility Option Price
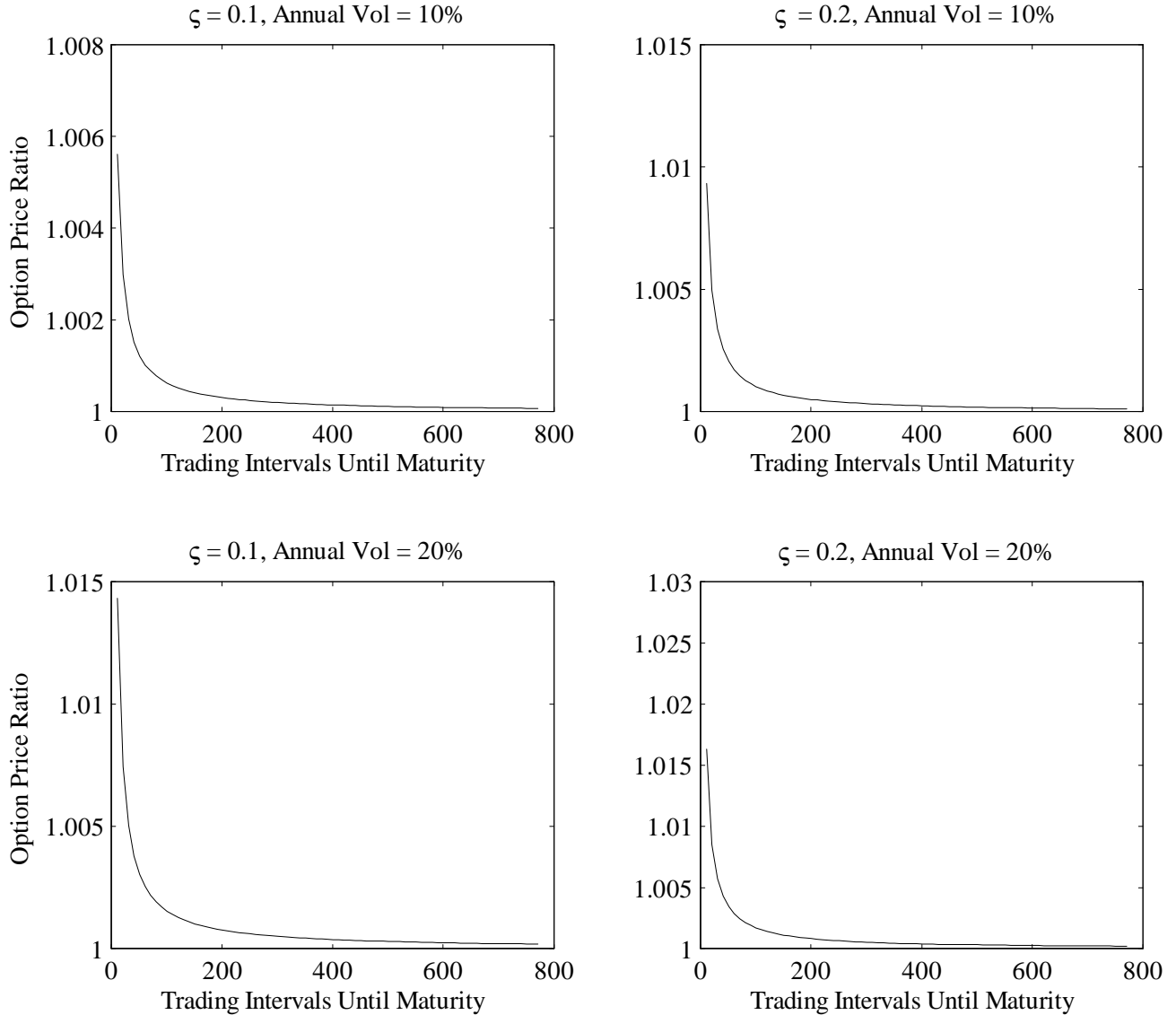
Notes to Figure: We plot the ratio of the Heston and Nandi (2000) discrete-time GARCH option price to the continuous-time SV option price in Heston (1993a) as the number of trading intervals until maturity gets large. We use $r = 0$, $K = 100$, $S = 100$, $M\Delta = 180$, $\kappa = 2$, and shock correlation $\rho = -1$. Return volatility is set to 10% per year ($v = \theta = .01$) in the top row and 20% in the bottom row ($v = \theta = .04$). The volatility of volatility parameter ς is set to 0.1 in the left column and 0.2 in the right column.

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